FEB 23 1939

VOLUME 25 FEBRUARY, 1939 NUMBER 2

PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

OF THE

UNITED STATES OF AMERICA

EXECUTIVE COMMITTEE

EDWIN BIDWELL WILSON, *Managing Editor*

RAYMOND PEARL R. G. HARRISON

PUBLISHED MONTHLY

Publication Office: Mack Printing Company, Easton, Pa.

Editorial Office: Harvard School of Public Health, Boston 17

Home Office of the Academy: Washington, D. C.

Entered as second-class matter June 21, 1920, at the post-office at Easton, Pennsylvania, under the act of August 24, 1912. Accepted for mailing at a special rate of postage, provided for in section 1103, act of October 3, 1917. Authorized June 18, 1920.

Price per annum, $5.00

OFFICERS OF THE ACADEMY

PRESIDENT.....F. R. LILLIE
VICE PRESIDENT.....A. L. DAY
FOREIGN SECRETARY.....L. J. HENDERSON
HOME SECRETARY.....FRED. E. WRIGHT
TREASURER.....ARTHUR KEITH

EDITORIAL BOARD

The HOME SECRETARY and the FOREIGN SECRETARY of the ACADEMY
The CHAIRMAN OF THE NATIONAL RESEARCH COUNCIL

V. BUSH	L. J. HENDERSON	R. A. MILLIKAN
R. E. COKER	F. B. JEWETT	H. R. MOODY
A. L. DAY	A. B. LAMB	W. J. V. OSTERHOUT
L. P. EISENHART	E. R. LONG	F. K. RICHTMYER
C. E. GUTHE	C. R. LONGWELL	F. H. SEARES
	W. R. MILES	

INFORMATION TO SUSCRIBERS

SUBSCRIPTIONS TO THE PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES at the rate of $5.00 per annum should be made payable to the National Academy of Sciences, and set to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C. Single numbers, $0.50.

PAST publications of the NATIONAL ACADEMY OF SCIENCES from 1863 to 1926 and of the NATIONAL RESEARCH COUNCIL from 1916 to 1925 are listed in the Index to the First Ten Volumes of these PROCEEDINGS. This Index appeared as Part 2 of Volume 13, No. 1, January, 1927, and is for sale separately at the price of $1.50. Orders should be sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C.

PAST volumes of the PROCEEDINGS may be obtained at five dollars per volume unless the sale of the volume would break a complete set of Volumes I to X. Single numbers may be obtained for fifty cents except where the sale of such numbers would break up a complete volume. Only two hundred complete sets of the PROCEEDINGS are available for sale—Volume I to X, price $75.00 (including Index, see *supra*). Orders should be sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C.

The following publications are issued by the NATIONAL RESEARCH COUNCIL. Orders and inquiries should be addressed: Publication Office, National Research Council, Constitution Ave. & 21st Street, Washington, D. C.

(A) THE BULLETIN OF THE NATIONAL RESEARCH COUNCIL published at irregular intervals. Individual numbers priced variously. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

(B) THE REPRINT AND CIRCULAR SERIES OF THE NATIONAL RESEARCH COUNCIL, individual numbers variously priced. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 25 February 15, 1939 Number 2
Copyright 1939 by The National Academy of Sciences

GROWTH AND DIFFERENTIATION IN LIVING PLANT MERISTEMS

By Edmund W. Sinnott

Department of Botany, Columbia University

Communicated January 12, 1939

Most of our knowledge of the cellular changes taking place during growth has been derived from a study of killed and sectioned material. Cell division in living plant tissues has been observed in a considerable number of cases, but chiefly in hairs, protonemata and similar structures which are only one or two cells thick. It is desirable to learn what takes place, in terms of changes in cell number, size and shape, in a typical living plant meristem as growth proceeds. Events cannot readily be observed here because most meristems are so massive as to be opaque. Furthermore, the terminal growing point of a stem is buried in surrounding structures, and that of a root is usually enclosed in a relatively opaque calyptra.

These difficulties may be overcome by the use of appropriate material and technique. Seedling roots growing from very small seeds are so delicate, often only six or eight cells thick, that light passes through them readily. In most cases, however, direct observation of the actual meristem is prevented by the root cap; but in some of the small-seeded grasses the cap is reduced to a group of only a few rather loose cells and the meristem is thus nearly naked. Species of *Agrostis*, *Phleum*, *Poa*, *Sporobolus* and related genera provide particularly favorable material.

It has been found possible to grow these very delicate roots under essentially normal conditions by germinating the seeds on strips of moist lens paper on microscopic slides, placed in covered staining dishes. About a centimeter of tap water is left in the bottom of the dish, which keeps the air humid and the paper moist. After about four days at room temperature the young roots are usually more than a centimeter long and are ready for study. The slide is now placed on the stage of a microscope and the root covered with a few drops of water. The meristem may be examined directly under a water immersion lens, but more satisfactory results have been obtained by placing a cover glass over the root, sup-

ported by glass chips on either side. The root may now be studied under any immersion objective, but the author has found water more satisfactory than oil for contact, since the latter tends to spread over the edges of the cover. Care must be used to avoid any pressure on the root when the cover glass is applied or removed, and to keep the root continually moist, for otherwise growth will stop immediately. By means of a camera lucida, drawings may be made of the cell outlines of the surface layer of cells, the second layer and sometimes the third layer. The slide should be returned to the damp chamber as soon as possible.

Drawings may be made at intervals of from one to several hours. If a marker, such as a cell of unusual shape or size, is located in the first drawing, then all the cells may be numbered with reference to it, and located again in the second and later drawings. In this way, all the new cell walls which have appeared through division in the interval may be recognized. Changes in cell size, both before and after division ceases, may also be measured. The actual cell lineage for a long series of cells from the apex of the meristem to the point where elongation has ceased may thus be determined. The nuclei are to be seen only vaguely and details of mitosis are not visible.

These roots are very sensitive to gravity and will not continue to grow normally more than a few minutes on the horizontal stage. It is more satisfactory in some respects to keep them vertical, with the microscope in a horizontal position, but this makes difficulty in the maintenance of liquid contact between lens and cover, and in the use of the camera lucida. Photographic records may be made, but unless conditions are very favorable, these have been found less satisfactory than drawings. The chief difficulty thus far encountered is the tendency for the growing tip to twist spirally and thus sometimes to prevent continuous observation of a given row of cells.

The approximate rates of cell division and of cell elongation may readily be measured. The rate of elongation of the whole root may also be accurately determined by marking the location of a given point under the camera lucida at constant time intervals. The characteristic rhythmic nutation of the root tip may be observed and its period and amplitude measured.

The method here described makes possible a fresh approach to the study of a number of other problems of growth and differentiation in a typical terminal meristem. Preliminary results in three of these problems, as studied in the genera *Poa*, *Phleum*, *Agrostis*, *Chloris* and *Sporobolus*, are presented briefly here.

1. *Factors Determining the Shape and Position of New Cell Walls.*—The cells in all layers are in regular longitudinal rows, so that the new walls, with rare exceptions, are transverse or essentially so. In equa-

tional divisions the walls are straight, but in non-equational ones, as in the divisions giving rise to root-hair initials in some genera, they are commonly curved from the first, with the convex side toward the larger cell. A new wall is never laid down directly opposite a previously formed cross wall of any neighboring cell. If the natural position for the new cross wall would tend to bring it into such a relation, this will be avoided by a bending of the new wall so that it joins the side wall at a point at one side or the other of the place where the cross wall of the neighboring cell is attached. A new cross wall tends to join the side wall halfway between the points of attachment of the two nearest cross walls, either of this cell or a neighboring cell. This is the position which would be assumed if the walls were liquid films. In many cases, however, this position would result in daughter cells very unequal in size, and in such cases the new wall is pushed considerably nearer the point of attachment of one of the cross walls. The point where the new cross wall joins the side wall thus seems to be determined by an equilibrium between the tendency to divide the mother cell equally (or into two parts of definite proportions) and the tendency to bisect the free side-wall space.

2. *"Sliding Growth."*—Since the point where each new cell wall is laid down can thus be definitely located with reference to the walls of all adjacent cells, and since this relationship can be followed throughout the history of all these cells, it is possible to determine with some certainty whether the cells slide along each other during the process of growth. This may best be studied in the relationships between the cells of the surface layer. All the evidence indicates that sliding growth does not occur. The relative positions of the transverse walls of adjacent cells are maintained essentially unchanged from the time they are laid down until the cells are many times their original size and growth stops. There is a definite gradient in cell expansion, beginning just before division ceases, and gradually increasing until maximum growth is attained. In this all the cells participate almost equally, though the root hair initials and certain stunted cells grow somewhat less rapidly than the rest. The relative position of their walls thus changes a little, but this is clearly due to a greater growth in some parts of the wall than in others and not to sliding growth. No case has been observed where the end wall of one cell has passed by the end wall of an adjacent cell during growth. The relative wall positions between the surface layer of cells and the layer immediately below have also been determined and here, too, the positions established when the cell walls are laid down persist to maturity.

3. *The Differentiation of Root Hairs.*—In *Phleum*, *Poa* and *Agrostis*, the cells which are to produce root hairs (the "trichoblasts" of Leavitt[1]) are definitely set apart at the last cell division. This is usually either the third or the fourth division from the tip of the root. At this division the

apical (distal) member of the pair of daughter cells is always smaller than the basal. It has denser protoplasmic contents and its walls tend to bulge outward slightly. At some distance back from the tip, when elongation has ceased, this cell produces a root hair. The other cell, which rarely may divide again, never produces a hair. In these genera, therefore, differentiation for this character occurs very early and the potencies of the cells are sharply limited almost from the beginning. In *Chloris* and *Sporobolus*, on the other hand, trichoblasts ordinarily do not appear, all the cells being essentially equal in size from the beginning and all being capable of producing root hairs. Some of the cells stain differently from others, and a differentiation of root hair cells may occasionally be observed before the hairs are formed. These plants evidently provide good material for a study of the factors controlling the differentiation of root hairs.

Summary.—Materials and methods are described, by the use of which it is possible to observe and measure the multiplication and growth of cells in living root meristems. This technique has been applied to the problems of the location of new cell walls, of "sliding growth," and of the differentiation of cells which are to form root hairs.

[1] R. G. Leavitt, *Proc. Boston Soc. Nat. Hist.*, **31**, 273–313 (1904).

QUANTITATIVE ANALYSIS OF THE INTERACTION OF INDIVIDUALS

BY ELIOT D. CHAPPLE

DEPARTMENT OF ANTHROPOLOGY AND DEPARTMENT OF INDUSTRIAL RESEARCH, HARVARD UNIVERSITY

Communicated January 13, 1939

Up to the present time, no quantitative studies of human interaction have been undertaken, with the exception of observations by Dorothy Thomas and her co-workers on the behavior of children.[1] These studies were designed to develop an "index of personality" based on the percentage of total time spent by a child handling objects, interacting with people or playing alone. Only percentage figures were secured for interaction, and the authors were more interested in developing criteria for determining the reliability of observers.

This paper represents a preliminary account of quantitative results secured through the use of a crude recording apparatus, now being superseded by an accurate instrument. Although the investigation was primarily exploratory, it is believed that a discussion of the results obtained by use of the old instrument will be of interest as an indication of the

possibilities of research in this field. It must be emphasized that the conclusions herein presented are entirely tentative. They must be tested by the use of accurate apparatus under laboratory conditions (using a one-way screen, etc.). This work is now in progress.

The present studies resulted from the development of operational methods for the description of the relations of individuals. These operations were evolved in the course of the analysis of material collected in field studies of communities. From the consideration of this material, a working definition of the class of phenomena was derived, which may be stated briefly as follows: Individuals are considered to be in interaction if the action of one individual is followed by the action of another individual. The full definition and a description of the analytic procedures are given elsewhere.[2] Here it is sufficient to point out that such an action is a manifest (hence observable) phenomenon, and may be made up of words, gestures, in general, of overt muscular activities. No distinction is made between kinds of actions.

In order to obtain a record of a sequence of actions manifested by individuals, it is necessary to use some kind of a time-recording apparatus. Accordingly, a simple device was improvised. A large wheel was fitted to the rubber roll of an old noiseless typewriter. A small electric motor drove this wheel by friction at a uniform rate of speed (15 inches to the minute). A roll of adding-machine paper, mounted on a brass frame, was fed through the roll to a take-up driven by another small motor. The take-up motor rewound the paper on a wooden roll and was arranged so that no pull was exerted on the paper coming through the typewriter. Thus, any influence on the speed due to the changing size of the upper roll was eliminated. An observer, seated at the typewriter, struck a designated key when the first individual acted, and when this action was ended by the action of the second individual, another key was struck. By the alternation of the two letters on the moving tape, a continuous record of a conversation could be secured. The length of time of each action was obtained by measuring from the top of the first letter to the top of the second. We thus obtained a series of durations of the actions of two individuals.

Observations were made only on two individuals at a time, due to the limitations of the machine. Other difficulties in recording, now eliminated in the new apparatus mentioned, were due to the inability to record those instances when two individuals talked at the same time. (In this case, individual B was recorded as ending A's actions, and then almost immediately A comes in again.) It was also impossible to mark where one individual fell silent in the midst of an action and began again when the other individual was not observed to manifest an action. Because of the lack of proper facilities for observation the observer had to be seated

in the same room, with the subjects at a distance of eight to ten feet away. The effect of this situation was to a large degree minimized, due to the fact that the subjects were colleagues who had known each other for several years, and the conversations were those which would ordinarily have taken place. The same observer was used in all observations described in this paper. For this reason, the error due to the observer may be regarded as constant, and is neglected in the rough approximations here described. We shall consider two series of eight observations, and one of five. One individual was a member of all three pairs. Due to difficulties in getting the subjects together, only a few observations were obtained on pairs made up of the other three individuals, and we shall therefore disregard those figures.

In each observation, lasting from thirty to fifty minutes, the two individuals manifested a total number of actions ranging from 320 actions (155 interactions) to 840 actions (420 interactions). The durations of any single action varied from 0.2 seconds (the minimum that could be recorded on the machine) to extreme values of well over a minute. Frequency distributions showed a characteristic asymmetry, the curve being definitely J-shaped. When one-second intervals were used, the mode fell in the interval between 0.2 and 1.1 seconds, containing about one-third of all values. The means of different individuals differ from observation to observation, the lowest 2.83, the highest 7.81, but approximately two-thirds of the values were found between the mean and 0.2, the other one-third in the higher values. In the present paper, we shall concern ourselves not with the probability that any single unit of action will have a given duration, but rather with the way in which the sequence of durations is arranged.

In order to describe the order of occurrence of the action-durations of two individuals, we obtained a kind of internal average by summing each five units of action. For reasons that need not be discussed here, attempts to use running averages were discarded. This summing of five should average out errors due to the observer; it is sufficiently small to be sensitive to fluctuations, yet at the same time affords the advantages of an average while enabling us to obtain the sum of the durations of any given series of actions. By comparison of the variation of the values of both individuals, A and B, we could begin to estimate the effect of one on the other. A visual representation was afforded by plotting the sequence of sums of fives for each individual on a graph, the ordinate representing the cumulative number of actions and the abscissa the cumulative time (sum of five) of the actions of each person. The same procedure was used in treating the units of interaction, summing by fives and plotting the results cumulatively.

Upon inspection of the curves for the separate individuals, it was seen

that no single empirical function described all the curves in a single series of observations. We therefore provisionally regarded the curves as made up of a series of straight lines, each describing that period of action during which an individual's rate was approximately constant. By selecting points on the curve in which the variation was a minimum, we calculated the slopes between the points and thus obtained a sequence of slopes for each individual. The rate of any individual for a given slope is defined by $\Sigma t/n$.

It should be pointed out that in the present study we have to regard these slopes as making up a sequence of rates of silence for each individual. This way of looking at the facts derives from the operations used to obtain the discrete durations. With the particular apparatus used, a typewriter with a moving tape, we are unable to obtain a measure of the duration of an action of one individual uninterrupted by the action of the other individual. Let us suppose that one individual is observed to act continuously without pause for twenty seconds. If the other person acts by nodding his head or uttering a monosyllable, this single action would be recorded as made up of three or four shorter actions. On the other hand, if one individual remains silent while the other individual continues to talk, the silent individual obviously cannot be interrupted. His silence will continue until he manifests an action. In other words, with this machine we could not distinguish the pauses an individual makes in an action during which the other person acts from those instances when both were talking at the same time. If the reader wishes to regard these figures as representing rates of action, he merely needs to shift the figures about. It should be pointed out, however, that this way of regarding the figures is not justified by the operations, and the results obtained (now to be described) reinforce this view.

In table 1, we present the slopes (mean rate), the sum of the durations, as well as the durations of each slope in chronological time for two conversations from different series. The table is arranged in order of the occurrence of these slopes in the sequence of units of interaction. From this table, it may be seen how one individual often maintains a slope while the other changes several times. There is no simple relation between the durations and the values of the slopes; nor is there any evident correlation between the slopes of one individual and the other.

In order to obtain an approximate description of the interaction of individuals represented by these slopes, we shall have to consider three questions: (1) what determines the value of the slope of an individual; (2) what determines the duration of any single slope of an individual; and (3) what is the nature of the dependence of the individuals upon each other; that is, is this relationship expressed in the associated slopes and their durations?

Let us first take up the question of the relationship of these slopes to one another. It is evident that we cannot express a single slope of individual *A* as a function of the slope of individual *B*, since we do not find

TABLE 1

SLOPES AND DURATIONS OF SLOPES IN TWO EVENTS

Observation No. 37

NUMBER OF ACTIONS IN SLOPE	*EDC* SLOPE	SUM OF *EDC*'S ACTION PER SLOPE	DURATIONS OF *EDC*'S SLOPE	*GB*'S SLOPE	SUM OF TIME *GB*'S ACTION PER SLOPE	DURATION OF *GB*'S SLOPE
0–10	6.11	61.1	93.5	3.65	54.7	131.4
10–15	2.49	62.3	437.6	--	--	--
15–25	--	--	--	29.01	290.1	318.4
25–35	--	--	--	6.17	339.1	761.3
35–55	9.69	193.9	303.8	--	--	--
55–70	3.68	55.2	157.7	--	--	--
70–75	21.66	108.4	132.1	--	--	--
75–80	7.73	773.0	1345.3	--	--	--
80–115	--	--	--	2.78	97.2	393.0
115–160	--	--	--	4.75	213.7	566.8
160–175	--	--	--	14.76	221.4	291.1

Observation No. 29

NUMBER OF ACTIONS IN SLOPE	*EDC* SLOPE	SUM OF *EDC*'S ACTION PER SLOPE	DURATIONS OF *EDC*'S SLOPE	*CMA*'S SLOPE	SUM OF TIME *CMA*'S ACTION PER SLOPE	DURATION OF *CMA*'S SLOPE
0–30	3.15	94.5	166.6	2.57	128.4	257.9
30–50	1.96	78.3	229.6	--	--	--
50–70	--	--	--	3.47	243.6	414.7
70–85	3.71	55.7	90.2	--	--	--
85–105	1.06	21.1	83.8	--	--	--
105–120	3.65	292.2	559.1	--	--	--
120–150	--	--	--	2.22	66.7	188.9
150–155	--	--	--	10.84	54.2	72.5
155–180	--	--	--	2.50	62.5	143.7
180–185	--	--	--	9.06	90.6	113.9
185–190	2.84	298.7	612.1	--	--	--
190–240	--	--	--	2.06	103.1	282.4
240–290	--	--	--	3.04	151.8	267.4

NOTE: The broken line, --, indicates that the above slope continues. It should be noted that *EDC* is characterized by a pattern made up of a slow slope followed by a fast one, a simple alternation, while *GB* has wide variations due to two very slow slopes, and *CMA* exhibits a fast-slow pattern but with intermediate pairs of values having a different ratio than the first and last pairs exhibited.

one slope of individual *A* associated solely with a single slope of individual *B*. In fact, such a situation is very rare in this material (five times in 333 slopes manifested); while one individual maintains a constant rate,

the other individual may manifest as many as five different slopes. If we hope to describe the duration of a single slope of individual A as some function of the slopes that B manifests during the time that A is constant we shall have to express the relationship as one between the durations. By estimating the ratio of the durations of A's slope to the total time (the sum of the durations of the slopes of both A and B), we can then see whether this varies as a function of the value of the slope.

When log slope is plotted as a function of this ratio for all the slopes manifested by one individual in interaction with another individual, it is found that the points representing the relationship make up a band. The band has a constant width and may be enclosed by two parallel lines

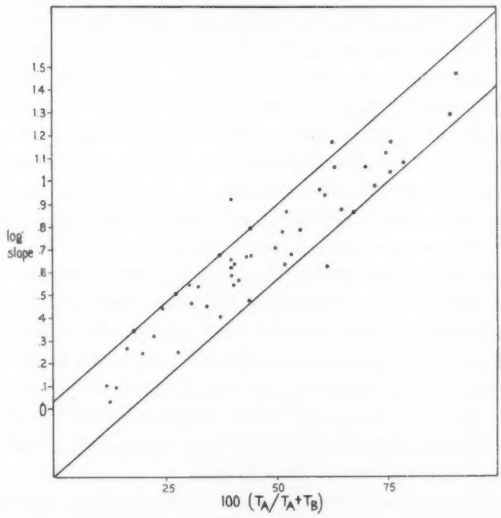

FIGURE 1

fitted to the points making up the border. Figure 1 represents the band of *GB* with *EDC*.

When the bands for the three pairs discussed here were plotted, it was found in the equation

$$\log S_A = k(t_A/t_A + t_B) + \text{constant}$$

where S equals the slope, and A and B are the two individuals, and t the sum of the durations, that the constant describing the slope of the band was the same for individual *EDC* in the three pairs in which he interacted, differing in slope from the bands of *GB*, *CMA* and *FLWR*, who also differed from each other. Unfortunately we were unable to obtain a sufficient

series for the other individuals with each other, but it is an obvious possibility that the constant k in this relationship is invariant for each individual. This similarity in the value indicates why these slopes should be regarded as representing silences. Since the same slope is found for the silences and not for the individual's actions in three separate series, it may be assumed either that the individual adapts the durations of his actions to the other person and does not adapt his silences, or that the recording device's limitations are responsible. No solution is possible, of course, until a more accurate apparatus is employed.

What precisely does the band function tell us? In the first place, it is evident that the function imposes limits upon the duration of any single slope. If individual A manifests a slope of a particular value, and B manifests a slope which does not bring the value of the ratio within the limits of the band at the given value of log slope for A, then individual A will maintain this slope, until other slopes manifested by B bring the ratio within the band limits. Thus, the duration of a slope at a given value is determined (within certain limits) by the proportion it makes with the total time. For the different individuals with whom *EDC* interacts, however, the band for the function has a different width, and it is suggested that the width of the band and the distribution of points within it is a measure of the degree of adaptation of the individuals to one another.

Not only do individuals seem to possess constant slopes with bands having definable limits; in one case, an individual, *FLWR*, exhibits a marked discontinuity. At a value of log 0.4800, the whole band shifts towards the X-axis and changes its slope, although the width of the band remains constant. This indicates, for this individual at least, that when he slows down to rates of silence above 3.02 seconds, the duration of his slope forms a greater proportion of the total time, and he has, therefore, differing rates of adjustment in interaction.

It should be pointed out that the limits of these bands cannot properly be regarded as defining the probability that repeated measurements of log slope as a function of the ratio will fall within the band instead of outside it. We shall come upon such a function a little later in this paper. In this case, however, the situation is more complex. If a band encloses the values of the function for each conversation, it will be observed that the width varies from day to day, although the slope remains constant. In some observations, all the points fall on a straight line; in others, points forming both borders of the total band are included in a single observation. The question of probability, then, involves the distribution of the variation for single observations. The limits of the band for all observations on a given pair define this probability. The view that the probability distribution is made up of bands for single observations is supported by the fact that the limits of the band do not act as limits which rigidly

control the change of slope. That is, when individual A changes to a certain slope, the first slope of individual B may bring the ratio within the band limits. That individual A does not shift depends upon two factors, (1) the existence of minimum and possibly maximum durations for slopes, the former giving a kind of threshold effect, the conditions for which have not yet been investigated; and (2) the limits of variability defined by the width of the band in a single observation. In this latter case, the variation in width of the bands is an expression of the variation in adjustment of the slopes of one individual to the slopes of the other, and hence depends upon the variability of the constituent individuals.

If we are to investigate this variability further, we must develop some way of dealing with the individual values of the slopes, regarding the preliminary definition of the problem of their durations as sufficient for our present purposes. In one-half of all observations, 13 out of 23 for *EDC* of which seven were with *CMA*, and a smaller number for the other individuals, the sequence of slopes forms a pattern of alternating fast and slow slopes. In the majority of these cases, there is a constant relationship between the fast and slow slopes. That is, if each sequence is broken up into a series of pairs, and each slow slope in a pair divided by its accompanying fast slope, the ratio or its first difference is constant for all pairs in the observation. The constant varies from observation to observation. If we assume provisionally that this alternation of fast and slow slopes is characteristic of individuals in interaction, we must then try to show why deviations from this pattern take place. We shall not attempt to determine the conditions defining the ratios of the slopes nor the variation in their absolute values when the ratios are constant.

If a consistent explanation is to be introduced to explain the deviations, we must find out whether the variability in the values of slopes differs for each individual or whether all the individuals under consideration vary in the same way. In order to obtain a measure of this variability, the dispersion of the slopes was expressed as a function of their mean value. If log σ slope is plotted as a function of log Mean slope, where $\sigma = \sqrt{\Sigma x^2/n - 1}$, the measurements for all four individuals fall within the limits of a narrow band defining the probability that measurements of σ as a function of the Mean will fall within the band instead of outside it.[3] The slope is greater than one and the origin is on the abscissa. Hence $\log \sigma = k \log M +$ Constant. It should be pointed out that a plot of σ of the raw figures against the Mean of the raw figures gives us a similar band with the slope approaching 1. It is, of course, premature to suppose that this function will be characteristic of all individuals on the basis of such a small sample, but it is of interest to point out that similar results have been obtained by Crozier and his co-workers for a wide variety of biological material.[4]

Since we know that these four individuals vary in exactly the same way, we do not have to regard the occurrence of extreme slopes for any single individual as due to a special property of that individual. Rather this variation or the causes thereof would seem to be common to all the individuals studied. If we assume that every individual alternates between fast and slow rates, then deviations from this pattern must have a consistent explanation. These deviations are uniformly very slow rates occurring either in a position where a slow rate was expected, but not one so large (on the theory that in any conversation, if the ratio of each fast slope to its accompanying slow slope is calculated, the constant derived is constant for all such pairs in a single conversation) or else such deviations occur in place of a fast slope, thus disturbing the regularity of the pattern. If the rates and durations of the other individual are then examined to see whether such deviations are preceded by uniform conditions, it is seen that in all cases the other individual has maintained a single slope for high numbers of interactions (from 30–100 units, the number varying with each individual). If it be remembered that the duration of a slope is determined (within limits) by the ratio of its duration to the total duration (sum of both A's and B's slopes during that time), then it is evident that deviations from the fast-slow pattern follow after a slope has been maintained *longer* than would have been the case if the rates of the two individuals had adapted to one another. This adaptation is, of course, determined by the value of the function, log slope $= k(t_A/t_A+t_B)+$constant, for the constituent individuals.

Summary.—In recording the interaction of pairs of individuals, the measurements secured represent the durations of the separate actions of an individual alternated with his silences or inactions. If each five actions of an individual are summed and plotted cumulatively, a series of slopes is obtained, during each of which the individual maintains an approximately constant rate.

The analysis of these slopes indicates that the duration of each slope is determined by the rate of the slope and its duration in chronological time (the sum of the durations of both A and B), the relationship having the form, log slope $= k(t_A/t_A+t_B)+$ constant, within the limits of a band of constant width. It is suggested that the constant is invariant for the individual.

Each individual is regarded as alternating between fast and slow rates. Although the values of these rates vary in a single conversation, the ratio of each fast rate to its accompanying slow rate is constant for the conversation, but varies from event to event.

Deviations from this pattern occur in more than half the cases. It is shown, however, that both representatives of the pattern (calculated) and its deviations vary in a uniform way. If the mean of the slopes of each

conversation is calculated and its standard deviation, it is found that log σ slope = k(log Mean slope) + constant, with all the points falling within a band of constant width. This relationship was the same for all observations on four individuals. It was then assumed that deviations could not be due to individual differences in variability as measured by this function.

It was then shown that deviations were in all cases preceded by slopes maintained over a period of time due to the operation of the suggested uniformity governing the duration of slopes at different values.

[1] Thomas, D. S., Loomis, A. M., Arrington, R. E., *Observational Studies of Social Behavior*, New Haven (1933).

[2] Chapple, E. D., *Measuring Human Relations* (to be published).

[3] Crozier, W. J., and Holway, A. H., *Proc. Nat. Acad. Sci.*, **24**, 3 (1938).

[4] Crozier, W. J., *Determinisme et Variabilite*, Paris, Hermann, 56 pp. (1935) (and citations to earlier work); *Jour. Gen. Physiol.*, **19**, 503 (1935–1936); Crozier, W. J., and Holway, A. H., *Proc. Nat. Acad. Sci.*, **24**, 3 (1938).

ASTROPHYSICAL CONSEQUENCES OF METASTABLE LEVELS IN HYDROGEN AND HELIUM

BY O. STRUVE, K. WURM AND L. G. HENYEY

YERKES OBSERVATORY, UNIVERSITY OF CHICAGO

Communicated January 9, 1939

1. The metastability of an atomic level can influence its population only under conditions which differ materially from thermodynamic equilibrium. In or near equilibrium the populations of the levels are independent of transition probabilities and depend only upon the temperature of the system. On the other hand, when the deviations from equilibrium are marked, as, for example, in the case of a gas excited by diluted black-body radiation (nebulae, outer shells of stars), cyclical processes play a fundamental rôle. Observationally the metastability of a level produces discernible effects when the interval of time between successive excitations is less than the lifetime of the level. Under these conditions, the metastable level can have a population of the same order of magnitude as in the case of equilibrium. Consequently, we may expect to observe in the continuous spectra of stars shining through the gas absorption lines arising from metastable levels, in addition to those arising from the ground level. Of course, for sufficiently great thicknesses of the gas we may observe weak absorption lines even when the metastability is not completely effective, provided that the number of atoms in the level is greater than about 10^{12} in a column whose cross-section is 1 cm.^{2} along the line of sight.

The existence of metastable levels in hydrogen (2S) and helium (2^1S

and 2^3S) is particularly interesting since the spectra of these elements can be observed astrophysically over an enormous range of conditions: from stellar atmospheres to nebulae and, in the case of hydrogen, even to interstellar space. The emission lines of H and He are very strong in the Orion Nebula.[1] In absorption, only the He line λ 3889, which arises from the metastable level 2^3S, is seen[2] superposed over the continuous spectra of stars which shine through the nebula. No H lines arising from the metastable 2S level have been found, in spite of the fact that H is almost certainly much more abundant than He. It is the purpose of this paper to show that this apparent discrepancy may be explained by a change in the lifetime of the metastable H level which depends upon the electron density of the gas.

For an excited level which connects directly with the ground level, the population in the presence of diluted radiation is:

$$n_3 = n_1 W e^{-\frac{h\nu_{13}}{kT}}, \tag{1a}$$

where W is the dilution factor. For a metastable level whose forbidden transition probability is A_{21} the population is[1]

$$n_2 = n_1 W \frac{A_{31}}{A_{21}} e^{-\frac{h\nu_{12}}{kT}}. \tag{1b}$$

2. The Balmer lines are very strong in emission in the Orion Nebula, and intensity measurements by Ambarzumian[1] indicate that the numbers of atoms in several quantum levels higher than the second are all of the order of 10^4 cm.$^{-2}$. The number of atoms in the second level is not known, because we cannot observe the Lyman emission lines and because observations fail to show any definite trace of the Balmer absorption lines. The latter are difficult to observe because all stars located in the Orion Nebula contain ordinary stellar absorption lines of hydrogen. However, in some of these stars the stellar Balmer lines are greatly broadened by the ionic Stark effect and by axial rotation. The absence of strong and sharp Balmer absorption lines of nebular origin is therefore well established. It is probable that if the number of atoms in the second level $n_2 \geq 10^{12}$, the absorption line of the nebula would have been observed. We shall assume that $n_2 < 10^{12}$. In the case of hydrogen we may assume that $\nu_{12} \approx \nu_{13}$. Hence from (1a) and (1b)

$$\frac{A_{31}}{A_{21}} = \frac{n_2}{n_3} < \frac{10^{12}}{10^4} = 10^8. \tag{2}$$

Accordingly

$$\tau_{2S} < 1 \text{ sec.} \qquad \text{(Orion Nebula)} \tag{3}$$

3. We have good evidence that interstellar space contains a considerable amount of atomic hydrogen, and a rough determination of the intensities of interstellar emission lines in certain regions of the Milky Way gives $n_3 = 10$ cm.$^{-3}$. This value is probably not representative of all regions of interstellar space, since it depends upon observations of selected regions in which the brightness of the emission lines is greater than the average. A better estimate for average conditions is

$$n_3 = 1 \text{ cm.}^{-3}.$$

This is the number of third-level atoms in a column having a cross-section of 1 cm.^{2} and extending to the effective limit of vision, which we estimate to be of the order of 1000 parsecs $= 3 \times 10^{21}$ cm. There is very good evidence from the spectrum of Nova Lacertae, whose distance is about 900 parsecs,[4] and from other objects, that there are no observable absorption lines of hydrogen. Hence $n_2 < 10^{12}$ and by formula (2):

$$\frac{A_{31}}{A_{21}} < 10^{12} \text{ and } \tau_{2S} < 10^4 \text{ sec. (Interstellar Space).} \qquad (4)$$

4. Bethe[5] has discussed theoretically the question of the lifetime of the $2^2S_{1/2}$ level of hydrogen. He states that the lifetime due to spontaneous transitions to the ground level is of the order of several months. If, however, the atom is in an electric field the atom can also pass to the ground level by first going to the $2^2P_{1/2}$ level. Indeed, under most astrophysical conditions the fields, due to the presence of ions, govern wholly the lifetime of this level. Bethe shows that, for a weak field, the ratio of the lifetime of the $2S$ level to the lifetime of the $2P$ level is given by

$$\frac{\tau_{2S}}{\tau_{2P}} = \left(\frac{2\beta}{k}\right)^2, \qquad (5)$$

where $1/4\beta$ is the lifetime of the $2P$ level and $k/2\pi$ is the Stark splitting of the $2S$ level, that is,

$$k = \frac{2\pi}{h}\frac{eF}{300} x_{2P}^{2S}. \qquad (6)$$

Here F is the electric field expressed in volts per centimeter and x_{2P}^{2S} is the matrix element of a coördinate of the electron in the hydrogen atom.

The average field due to the presence of electrons has also been considered by Bethe. If n is the number of electrons per cm.^{3}, the electron occupies a volume $1/n$ which, if replaced by a sphere, has a radius

$$r = \left(\frac{3}{4\pi n}\right)^{1/3}. \qquad (7)$$

Bethe takes one-half of this radius as the mean distance between the hydrogen atom and the nearest electron. The field of the latter is, therefore,

$$F = 300\, e \left(\frac{2}{r}\right)^2 \text{ volts/cm.} \tag{8}$$

The fields due to other electrons can be neglected since they can be regarded as forming a uniform charge distribution.

Equations (5) to (8) can be combined so as to express the ratio (5) in terms of n and of atomic constants. Inserting the numerical values of the constants,[6] we find:

$$\frac{\tau_{2S}}{\tau_{2P}} = 2.3 \times 10^{14} n^{-4/3}. \tag{9}$$

Since[7] $\dfrac{1}{\tau_{2P}} = 4\beta = 6.3 \times 10^{-8}$, we have

$$\tau_{2S} = 4 \times 10^{6} n^{-4/3}. \tag{10}$$

It is also of interest to investigate the order of magnitude of this effect for the metastable level 2^1S of helium. In a field the 2^1S level interacts with the 2^1P level and becomes partly a P level. The ratio of the lifetimes of the 2^1P and 2^1S levels are in the ratio of the Stark splitting to the separation of the interacting levels.[8] Using hydrogenic wave functions, Foster[9] has computed the Stark splitting for a weak field as $0.0285\ F^2$ cm.$^{-1}$, where F is expressed in 10^5 volts/cm. The separation of the levels is 5857 cm.$^{-1}$. Hence, expressing F again in terms of n, we have

$$\frac{\tau_{2^1S}}{\tau_{2^1P}} = 9 \times 10^{26}\, n^{-4/3}.$$

A comparison of this equation with (10) shows that the effect under consideration is of a completely different order of magnitude for the two cases. The importance of the effect for hydrogen is due to the fact that the $2S$ and $2P$ levels have exactly the same energy. For the metastable 2^3S level of helium the effect of a field is even smaller than for the 2^1S level. Actually, under most conditions, the lifetime of the 2^1S level will be governed by other factors, particularly by collision transitions to the 2^3S level.

5. For interstellar space we have, approximately:[3]

$$n_e = 30 \text{ cm.}^{-3}.$$

Hence, formula (10) gives

$$\tau_{2S} = 4 \times 10^{3} \text{ sec. (Interstellar Space).} \tag{11}$$

Before we compare this value with the observational result (4), we must

multiply the latter by 4 to take account of the fact that the levels $2P_{1/2}$ and $2P_{3/2}$ are not metastable. We conclude that the theoretical lifetime is consistent with the observational value.

For the Orion Nebula n_e is not known. Hence, we determine n_e from (10) in such a manner that (3) is obeyed:

$$n_e > 2 \times 10^4 \text{ cm.}^{-3}.$$

We have no means to verify this result by direct observation. For the Trifid Nebula, in which the density is probably considerably smaller than in the Orion Nebula, Rosseland[10] found

$$n_e = 235 \text{ cm.}^{-3}.$$

The somewhat higher value for the Orion Nebula appears entirely plausible.

6. O. C. Wilson[2] has found that in the Orion Nebula He I (2^3S–3^3P) λ 3889, appears as a strong absorption line. No accurate determination of the equivalent width is available, but we estimate $n_2 = 10^{13}$ cm.$^{-2}$. The emission lines of He I are strong, but no determinations of n_3 have been made. Perhaps we may estimate $n_3 = 10^3$ cm.$^{-2}$. This would give from (2):

$$\frac{A_{31}}{A_{21}} = 10^{10} \text{ and } \tau_{2^3S} = 10^2 \text{ sec. (Orion Nebula).} \qquad (12)$$

Accordingly, the metastability of the He I 2^3S term in the Orion Nebula is much more pronounced than that of the H $2S$ term. This is in good agreement with the theory because we have seen that the theory would give an exceedingly long lifetime for the term.

The question arises whether the astronomical evidence is compatible with a practically infinite lifetime of the 2^3S level, as has been suggested[11] from physical theory. Apparently this is not the case. Formula (1b) holds if $W \ll \frac{A_{21}}{A_{31}}$. If $W \gg \frac{A_{21}}{A_{31}}$, we have approximately:

$$n_2 = n_1 e^{-\frac{h\nu}{kT}}. \qquad (13)$$

If this were true (13) and (1a) would give

$$\frac{n_3}{n_2} = W = 10^{-10}.$$

It is probable, however, that for those parts of the Orion Nebula to which the observations refer W is more nearly of the order of 10^{-14} or 10^{-15}. Hence, we should be inclined to regard (12) as real.

The He I line (2^1S–4^1P) λ 3965 has been looked for in the spectra of the

same Orion stars which show strong (2^3S–3^3P). No absorption line has been found. However, since all lines of the singlet system of He are much weaker than the corresponding lines of the triplet system, we cannot assert, from the observational evidence, that the lifetime of the 2^1S level is necessarily shorter than that of the 2^3S level. A careful search for the line λ 5016 should settle this question.

7. The spectrum of Nova Lacertae shows conclusively that for interstellar space the He I line (2^3S–3^3P) gives $n_2 < 10^{12}$. Since no emission lines have been observed we cannot determine the upper limit of A_{31}/A_{21}. The ionization of the interstellar gas is much lower[3] than that of the Orion Nebula; hence the absence of the He I lines is not surprising.

8. Formula (10) shows that for relatively low electron pressures the $2S$ state of H completely loses its metastability. Thus, when

$$n_e \approx 10^{10} \text{ cm.}^{-3}$$

the lifetime becomes of the order of 10^{-8} sec. This is of the order of the electron density in the chromosphere. In outer shells of B and O stars, where the electrons come almost entirely from the ionization of hydrogen, we estimate $n_e = 10^6$ cm.$^{-3}$. Hence the hydrogen lines should behave as though they originated from a metastable level.[12] However, it is possible that in some relatively dense shells n_e becomes large enough to reduce the lifetime of the $2S$ state of H to such an extent that the Balmer lines will lose with respect to lines of other elements which originate from the ground level or from a metastable level. One is reminded of the great strength of the He I line λ 3889 in γ Cas[13] which surpasses the Balmer lines, although in normal giant spectra (where Stark effect is absent) the Balmer lines are always much stronger than the He lines. High density and strong ionization reduce the lifetime. Hence, in late-type giants, conditions are favorable for a high degree of metastability of the $2S$ level. The enhancement in α Ori of the multiplets (a^5D–z^7D°) and (a^5D–z^7P°), which originate from the ground level of Fe I, show that departures from thermodynamic equilibrium are conspicuous. It is tempting to explain the great strength of the Balmer absorption lines in supergiants of late spectral type as a consequence of the metastability of the $2S$ level. However, it is premature to discuss this matter further until accurate measurements of the Paschen lines have been made.

[1] V. Ambarzumian, *Zeits. Astrophys.*, **6**, 112 (1933); *Poulkovo Obs. Circ.*, **6** (1933). This expression is valid only when W is small compared to A_{21}/A_{31}.

[2] O. C. Wilson, *Pub. Astr. Soc. Pacific*, **49**, 338 (1937).

[3] Struve, *Proc. Nat. Acad. Sci.*, **25**, 36–43 (1939).

[4] J. A. Pearce, *Pub. Astr. Soc. Pacific*, **49**, 149 (1937).

[5] *Handbuch der Physik*, **24**, Pt. 1, 452 (1933).

[6] The matrix element $x^{2^2S}{}_{2^2P}$ is $\sqrt{3}$ times the Hartree unit of length, as may be verified from the formulae 34.2, et seq., of Bethe, into which we insert $n = 2$, $j = 1/2$ and $m = 1/2$.

[7] Bethe, *op. cit.* Table 17, page 144.

[8] Condon and Shortley, *The Theory of Atomic Spectra*, p. 413 (1935).

[9] *Proc. Roy. Soc.*, **A117**, 137 (1937).

[10] *Theoretical Astrophysics*, p. 319 (1936). In his earlier book, *Astrophysik auf Atomtheoritischer Grundlage*, p. 235 (1931), Rosseland gave $n_e = 470$ cm.$^{-3}$.

[11] We are indebted for stimulating discussions of this point to Dr. D. H. Menzel of Harvard University and Dr. G. Breit of the University of Wisconsin.

[12] Struve and Wurm, *Astrophys. Jour.*, **88**, 107 (1938).

[13] R. Baldwin, *Astrophys. Jour.* (in press). See spectrum (a) of his illustration.

THE RESOLUTION OF SIX TESTS INTO THREE GENERAL FACTORS

BY EDWIN B. WILSON AND JANE WORCESTER

HARVARD SCHOOL OF PUBLIC HEALTH

Communicated January 14, 1939

It has been pointed out that if n tests are to be resolved into l generals and n specifics the expression

$$T = \frac{1}{2}(n - l)^2 - \frac{1}{2}(n + l)$$

must be zero or negative and that when $T = 0$ the resolution if possible at all is unique in the sense that no arbitrary parameter is involved in the solution but not necessarily in the sense that there shall be only one solution.[1] The case of six tests and three factors is the simplest in which the possibility of multiple distinct solutions arises. It is the object of this note to exhibit a 6 by 6 correlation matrix which has two solutions for the resolution.[2]

The problem is to determine six communalities h_i^2 so that the matrix

$$\left\| \begin{matrix} h_1^2 & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} \\ r_{12} & h_2^2 & r_{23} & r_{24} & r_{25} & r_{26} \\ r_{13} & r_{23} & h_3^2 & r_{34} & r_{35} & r_{36} \\ r_{14} & r_{24} & r_{34} & h_4^2 & r_{45} & r_{46} \\ r_{15} & r_{25} & r_{35} & r_{45} & h_5^2 & r_{56} \\ r_{16} & r_{26} & r_{36} & r_{46} & r_{56} & h_6^2 \end{matrix} \right\|$$

shall reduce to rank 3. There is no four-rowed minor which does not contain at least two communalities; consider therefore

$$\begin{vmatrix} r_{13} & r_{14} & r_{15} & r_{16} \\ r_{23} & r_{24} & r_{25} & r_{26} \\ h_3^2 & r_{34} & r_{35} & r_{36} \\ r_{34} & h_4^2 & r_{45} & r_{46} \end{vmatrix} = 0.$$

This expands into a quadratic equation in the communalities h_3^2 and h_4^2 as

$$h_3^2h_4^2\begin{vmatrix} r_{15} & r_{16} \\ r_{25} & r_{26} \end{vmatrix} + h_3^2\begin{vmatrix} r_{14} & r_{15} & r_{16} \\ r_{24} & r_{25} & r_{26} \\ 0 & r_{45} & r_{46} \end{vmatrix} + h_4^2\begin{vmatrix} r_{13} & r_{15} & r_{16} \\ r_{23} & r_{25} & r_{26} \\ 0 & r_{35} & r_{36} \end{vmatrix} + \begin{vmatrix} r_{13} & r_{14} & r_{15} & r_{16} \\ r_{23} & r_{24} & r_{25} & r_{26} \\ 0 & r_{34} & r_{35} & r_{36} \\ r_{34} & 0 & r_{45} & r_{46} \end{vmatrix} = 0$$

Including the above, three equations may be obtained by non-equivalent permutations among the subscripts 1, 2, 5, 6; these three are linearly related, indeed if their signs be taken aright their sum is 0; of the set of three, however, any pair are usually independent and may be solved for h_3^2 and h_4^2 to give a pair of solutions, ordinarily different. When either of these pairs is inserted in the place of h_3^2 and h_4^2, four-rowed minors may be found which contain only one other communality and hence the complete solution contains just two sets of communalities.

The following matrix

$$\begin{Vmatrix} h_1^2 & .56 & .16 & .48 & .24 & .64 \\ .56 & h_2^2 & .20 & .66 & .51 & .86 \\ .16 & .20 & h_3^2 & .18 & .07 & .23 \\ .48 & .66 & .18 & h_4^2 & .30 & .72 \\ .24 & .51 & .07 & .30 & h_5^2 & .41 \\ .64 & .86 & .23 & .72 & .41 & h_6^2 \end{Vmatrix}$$

may be reduced to rank 3 by either of the following set of communalities:

$$h_1^2 = .64 \text{ or } h_1^2 = \frac{432}{1015} = .425616$$

$$h_2^2 = .85 \text{ or } h_2^2 = \frac{1173}{1300} = .902308$$

$$h_3^2 = .06 \text{ or } h_3^2 = \frac{311}{4900} = .063569$$

$$h_4^2 = .56 \text{ or } h_4^2 = \frac{711}{1300} = .546923$$

$$h_5^2 = .50 \text{ or } h_5^2 = \frac{116}{300} = .386667$$

$$h_6^2 = .93 \text{ or } h_6^2 = \frac{998}{1000} = .998000$$

It will be observed that although some of the communalities in the two sets are nearly equal, others, notably h_1^2 and h_5^2 (and, presumably, because of their nearness to 1, h_6^2) are decidedly different. It may further be noted

that in both sets some of the communalities are tolerably far removed from the value of the largest correlation coefficient in their row or column which is the provisional value assigned by Thurstone in his centroid method of determining the factor loadings.[3]

One set of factor loadings for the two solutions is:

	I	II	III	I′	II′	III′
x_1	.8	0	0	.6524	0	0
x_2	.7	.6	0	.8584	.4068	0
x_3	.2	.1	−.1	.2453	−.0259	.0515
x_4	.6	.4	−.2	.7358	.0699	.0265
x_5	.3	.5	.4	.3679	.4774	−.1530
x_6	.8	.5	−.2	.9810	.0441	.1835

It will be noted that the factor loadings are different, often decidedly different. Furthermore, although the set I, II, III or the set I′, II′, III′ may be changed by an orthogonal transformation of ∞^3 degrees of freedom operating upon them, the two sets cannot be obtained one from the other by any orthogonal transformation operating upon them[4]—the solutions are distinct.

If we should take the correlation matrix

$$\begin{Vmatrix} h_1^2 & .56 & .16 & .48 & .24 & .64 \\ .56 & h_2^2 & .20 & .66 & .51 & .86 \\ .16 & .20 & h_3^2 & .08 & .27 & .13 \\ .48 & .66 & .08 & h_4^2 & .30 & .72 \\ .24 & .51 & .27 & .30 & h_5^2 & .41 \\ .64 & .86 & .13 & .72 & .41 & h_6^2 \end{Vmatrix}$$

and try to determine the communalities so as to reduce the rank of the matrix to 3 we should find only one set of solutions all of which lay between 0 and 1 and hence were possible; the other set, though real numbers, would contain at least one communality which was either negative or greater than 1. The solution for the possible set of communalities and of factor loadings is

		I	II	III
h_1^2	.64	.8	0	0
h_2^2	.85	.7	.6	0
h_3^2	.21	.2	.1	.4
h_4^2	.56	.6	.4	−.2
h_5^2	.50	.3	.5	.4
h_6^2	.93	.8	.5	−.2

If we should take the correlation matrix

$$\begin{Vmatrix} h_1^2 & .56 & .16 & .48 & .24 & .64 \\ .56 & h_2^2 & .20 & .66 & .51 & .86 \\ .16 & .20 & h_3^2 & .18 & .07 & .25 \\ .48 & .66 & .18 & h_4^2 & .30 & .76 \\ .24 & .51 & .07 & .30 & h_5^2 & .33 \\ .64 & .86 & .25 & .76 & .33 & h_6^2 \end{Vmatrix}$$

we should find that no set of communalities could be obtained which would reduce the rank to 3 and hence permit a resolution into 3 generals and 6 specifics. The three matrices do not look notably different.

If the pentad expression[5] be defined as

$$\begin{aligned} P_{12345} = {} & r_{12}r_{13}r_{24}r_{35}r_{45} + r_{14}r_{15}r_{23}r_{24}r_{35} + r_{12}r_{14}r_{25}r_{34}r_{35} \\ & + r_{13}r_{15}r_{24}r_{25}r_{34} + r_{12}r_{15}r_{23}r_{34}r_{45} + r_{13}r_{14}r_{23}r_{25}r_{45} \\ & - r_{12}r_{14}r_{23}r_{35}r_{45} - r_{13}r_{15}r_{23}r_{24}r_{45} - r_{12}r_{15}r_{24}r_{34}r_{35} \\ & - r_{13}r_{14}r_{24}r_{25}r_{35} - r_{12}r_{13}r_{25}r_{34}r_{45} - r_{14}r_{15}r_{23}r_{25}r_{34} \end{aligned}$$

the quadratic equation for any communality, say h_3^2, becomes

$$P_{12456}(h_3^2)^2 + H_{3\cdot12456}h_3^2 + K_{3\cdot12456} = 0,$$

where H and K, like P, are skew symmetric in the letters 1, 2, 4, 5, 6 and are in fact as follows:

$$\begin{aligned} K_{3\cdot12456} &= r_{31}^2P_{32456} + r_{23}^2P_{34561} + r_{34}^2P_{35612} + r_{35}^2P_{36124} + r_{36}^2P_{31245} \\ H_{3\cdot12456} &= r_{31}r_{32}[r_{45}r_{56}(r_{14}r_{26} - r_{16}r_{24}) + r_{46}r_{45}(r_{16}r_{25} - r_{15}r_{26}) + \\ &\qquad r_{46}r_{56}(r_{15}r_{24} - r_{14}r_{25})] \end{aligned}$$

+ *nine similar terms in which the pair* 1, 2 *is replaced by any other pair of two from the five numbers* 1, 2, 4, 5, 6.

The analytical conditions for resolubility into two sets or just one set or no set of three generals and six specifics are naturally complicated; it is probably better to proceed to effect a solution arithmetically.

If one should go up to the next case in which the resolution may be "unique," viz., $n = 10, l = 6$, there may well be a higher multiplicity than two for distinct solutions.

If one should undertake to find solutions by Thurstone's centroid method it is clear that some method of successive approximations and convergence would be necessary—and necessarily very tedious.[6] It may well be that the actual determination of any or all solutions in a particular case would be of no value, but it does seem to be of some value to recognize that the method of factor analysis as a mathematical method admits of a plurality of solutions in some cases. In so far as three general factors are deter-

minable from six tests there may logically be two distinct solutions—and therewith two distinct sets of specifics but the team tests for corresponding specifics in the two sets must be the same.[7]

[1] E. B. Wilson, *Jour. Gen. Psychol.*, **2**, 153–172 (1929), especially p. 156.

[2] The authors are indebted to the Carnegie Corporation for a grant to cover the cost of the calculations involved.

[3] L. L. Thurstone, *The Vectors of Mind*, p. 89. As a matter of fact h_3^2 in either set is less than any correlation coefficient in its row or column, and h_6^2 is greater than any.

[4] Of course, as the two sets are each three mutually orthogonal unit vectors, either can be carried into the other by some orthogonal transformation operating not in the 3-space defined by either set but in the 6-dimensional space defined by the six vectors of the two sets—this being not the 6-space defined by the 6 tests.

[5] T. L. Kelley, *Crossroads in the Mind of Man*, p. 58.

[6] If one should apply Thurstone's method without proceeding to successive approximations one would find a set of generals, with their factor loadings, which was not identical with either of the two sets I, II, III or I′, II′, III′. If it were true that the set found could be rotated into either of these sets by an orthogonal transformation operating upon it, the lack of identity might be of no significance by virtue of Thurstone's assumption that in any case the set found is without psychological significance until it has been so rotated as to be psychologically interpretable; but as the solution found by Thurstone's method is not in general in the 3-space of I, II, III or of I′, II′, III′ no rotation of it in the sense in which he uses that word will bring it into agreement with either of those sets which the general theory indicates might be of significance. It may well be that the Thurstone centroid solution and the true distinct solutions I, II, III and I′, II′, III′ all lie nearly enough in the same 3-space to be practically indistinguishable when one has regard to the sampling errors of the correlation coefficients determined from relatively small samples. It may also well be that within the freedom permitted by those sampling errors the question as to whether the resolution can be made in two, one or no ways is itself indeterminate. If more than 6 tests are to be resoluble into three general factors plus specifics very complicated analytical conditions must be satisfied by the correlation coefficients. If 6 tests are resoluble into three general factors plus specifics in two ways one may conceivably build up additional tests which would be consistent either with the solution I, II, III or with the solution I′, II′, III′, but ordinarily not with both. We may thus have two sets of n tests, $n > 6$, resoluble into three generals and n specifics, the generals as well as the specifics being different, even though there be six tests common to the two sets.

[7] E. B. Wilson, *Proc. Nat. Acad. Sci.*, **20**, 193–196 (1934).

TEMPERATURE AND THE CRITICAL INTENSITY FOR RESPONSE TO VISUAL FLICKER. II

BY W. J. CROZIER

BIOLOGICAL LABORATORIES, HARVARD UNIVERSITY

Communicated January 13, 1939

1. Initial tests of the relation between temperature and the curve of flash-frequency (F) vs. flash-intensity (I) critical for response to flicker[1] showed that change of temperature merely shifts the $F - \log I$ contour on the $\log I$ axis, without essentially altering its form. Rise of temperature moves the curve to lower critical intensities. This fact is of primary significance for the theory of the flicker-response contour.[2] The exact mode of dependence of the shift in position upon temperature, as revealed by more elaborate experiments with various animals, thus assumes particular importance. The first experiments[1] were made at 3 temperatures only. For an insect (*Anax*, nymph) and a teleost (sunfish, *Enneacanthus*) it appeared that a somewhat unusual relationship obtains. Theory calls for the reciprocal of the critical intensity at a fixed flash-frequency as the proper measure of excitability. The temperature characteristic[3] (μ in the Arrhenius equation) for $1/I$ was not constant but *increased* with rise of temperature. Hence it was suggested[1] that visual excitability measured in this way might well be so complexly determined as to make its analysis quite involved.

More complete investigation was made of the flicker-response contour of the turtle *Pseudemys*.[4] Here a simplex curve is found, without complications due to the gross morphology of the eye (as in most arthropods[5]) and without the involvement of two sets of visual response-elements ("rods" and "cones" as in most vertebrates). As fundamentally in all other cases investigated, the $F - \log I$ contour is a probability integral. The adequacy of this formulation arises directly from the fact that a large number of elements are available for the determination of the response, and that each of these fluctuates in its contribution. Elevation of temperature reduces the critical intensity for each of these elements in the same proportionate way, if the population is homogeneous with respect to type of chemical organization. Hence if excitability is governed in the fashion typical for physiological processes involving speeds or frequencies,[6] we expect $1/I$ (F constant) to obey the Arrhenius equation, with constant μ's. For *Pseudemys* this was found.[4]

The exceptional instances first encountered were therefore reinvestigated with especial care. Processes demonstrably complex in their quantitative dependence on temperature should provide significant tests of the general theory of temperature characteristics. Of these the most important for

the view that the control of speeds and frequencies in biological systems is essentially simple rather than complex, are those instances in which μ increases steadily with rising temperature. Processes of this kind are necessarily governed by the resultant of 2 or more different concurrent reactions with different "activation energies" simultaneously contributing to the control of the occurrence of the observed result.[7,8] Such processes have seldom been established in biological observations—a fact which is in itself significant. The meaning of the data in the single most fully illustrative case, involving the respiration of sea urchin eggs,[9] has been completely missed.[10]

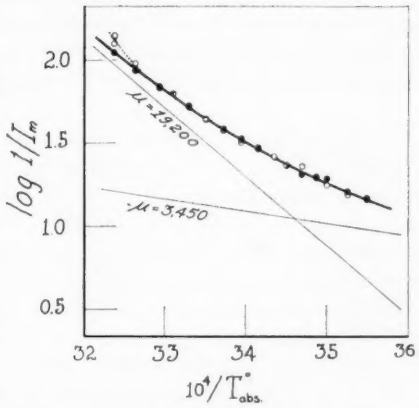

FIGURE 1

$1/I_m$ at $F = 20$ (solid dots) and at $F = 55$, at various temperatures, for *Anax* nymphs, with light-time fraction = 0.50. Each I_m is the mean of 30 measurements.[11] Values for $F = 55$ are multiplied by antilog 1.205. The exceptional points at $F = 55$, 35°.8 are by experimental analysis known to be due to special effects at this level;[11] I_m departs by only 0.03 to 0.09 log unit. The upward concavity on the Arrhenius grid shows that μ in the equation $1/I = exp.\ (-\mu/RT)$ increases continuously as T rises. Consequently at least 2 different processes participate concurrently in the control of $1/I$. The curvature found is accurately reproduced by the summation of the two processes whose velocity curves are shown below, one with $\mu = 19{,}200$ and the other with $\mu = 3400$; their velocities are equal at 15.9°.

2. Determinations of critical intensity for response of *Anax* nymphs to visual flicker have been made at 2 flash-frequencies ($F = 20/sec.$ and $F = 55$) at temperatures between 8° and 36°.[11] The technique, methods of calculation and tests for statistical homogeneity of the material have been described elsewhere.[11,12] The form and maximum of the $F - \log I$ curve is independent of temperature. (A minor complication not significant for the present discussion occurs at 36° for $F = 55$ only.[11]) The apparent temperature characteristic for $1/I$ is the same for $F = 20$ and $F = 55$, and increases steadily as $t°$ increases (Fig. 1).

The upwardly concave curve in figure 1, with log $(1/I)$ plotted against $1/T°_{abs.}$, is easily accounted for as the resultant of the concurrent action of 2 processes with $\mu = 19{,}200$ and $\mu = 3400$, respectively, with velocities equal at 15.9° (Fig. 1). The suggestive correspondences of $\mu = 19{,}200$ for dehydrogenations[11] (as of sugars) and such processes as diffusion for

the low μ, point the way for specific attempts at the identification of these processes through suitably designed experiments.

3. The flicker contour for the sunfish *Enneacanthus* is found to present a different kind of situation.[13] Its elucidation again illustrates the necessity for closely spaced observations of known relative precision for the interpretation of temperature functions. With *Enneacanthus* there occurs a break in the $1/I$ curve for a given F, at *ca.* 20°C. Above this temperature μ is *higher* than from 8° to 20°. Relations of this type are uncommon, but not unknown,[8,14] and there is nothing in theory which forbids them.

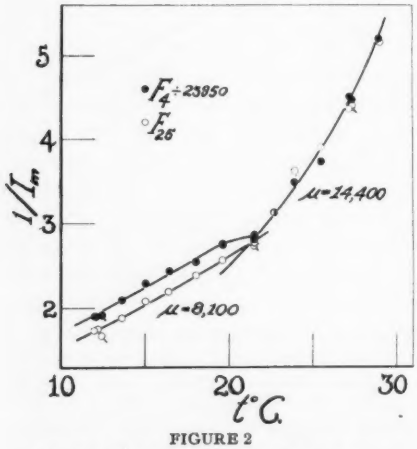

FIGURE 2

$1/I_m$ for $F = 4$ and $F = 20$, the former divided by 23,950, with the sunfish *Enneacanthus*. Each point is the mean of 30 observations. Points with tags are from an early series.[1] For the "rod" ($F = 4$) and the "cone" ($F = 20$) parts of the $F = \log I$ curve the μ's are identical. The critical temperature (20°) is marked by a wider dislocation of the $F = 4$ measurements.

One flash-frequency ($F = 4$) was chosen in the so-called "rod" section and another ($F = 25$) in the "cone" part of the sunfish flicker contour.[15] The *shape* of the $F - \log I$ curve is independent of temperature, in each of these segments. But the break in the temperature curve (Fig. 2) is unmistakably emphasized by the fact that for the "rod" part of the function the curve is definitely shifted downward on the $1/I$ axis to a lesser extent than for the cone part, below 20°. This makes it obvious that the critical temperature at *ca.* 20° is a real phenomenon in this system, even if it were not already clear that no single continuous curve will adequately describe the data at $F = 25$ (Fig. 2).

The values of μ for the sunfish data are 8100, 14,400. The same values hold for both "rod" and "cone" sections of the curve. At any temperature the values of $1/I_m$ for $F = 4$ and $F = 25$ are in the ratio of 24,000:1 or (below 20°) more. This is a rather striking illustration of the fact that μ for a biological system is essentially independent[8,14,16] of the *velocity* of the underlying process (here measured by $1/I$).

4. That diverse magnitudes of μ are encountered for flicker-excitability in different animals naturally signifies that the specific chemical control of the excitability is different. This does not necessarily mean that the

substances undergoing chemical change as result of illumination are different. Substrate materials in the excitability reactions could be the same, but catalysts different.

It is just as likely, however, that the substances involved could even be the same, and their metabolic interrelations, but that in different kinds of animals different steps or links in the system are found in the slow "pacemaker" rôle.[4] The data on experimental modification of μ[17] make this entirely possible. Differences in the locations of critical temperatures (*Pseudemys*, 20°; *Enneacanthus* 30°) prove that such differences in organization are real.

Application of the hypothesis[18] that the data of visual flicker are to be interpreted in terms of retinal photochemistry has led[19] to the conclusion that for *Enneacanthus* there must be deduced a different photochemical stationary state equation for the "rod" and "cone" parts of the contour. For the "rods," light and dark processes are each of second order, for the "cones," light process is first order, dark process second. The equation does not in fact describe the $F - \log I$ curve,[20] and is moreover inconsistent with the direction of the temperature shift; apart from this, the invariance of μ for "rods" and "cones" (Fig. 2) shows directly that the chemical control of excitability is the *same* in both.

This is not in conflict with the existence of 2 populations[21] of sensory effects, as proved by their behavior in inheritance[22] and by the present results with *Enneacanthus*. Since there is every reason to regard these 2 populations of elementary sensory effects as resident in the central nervous system,[23] there is a general basis for expecting their chemical organization to be of the same kind. That it is of the same kind, in *Enneacanthus*, is shown by the fact that with increasing proportion of light-time in the flash-cycle the rates of increase of the abscissa of inflection in the "rod" and "cone" parts of the $F - \log I$ curve are identical.[24]

[1] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **20**, 393, 411 (1936–1937).

[2] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Ibid.*, **20**, 411 (1936–1937); **21**, 17 (1937–1938).

[3] Crozier, W. J., *Ibid.*, **7**, 123 (1924–1925).

[4] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Proc. Nat. Acad. Sci.*, **24**, 125, 216 (1938); *Jour. Gen. Physiol.*, **22** (1938–1939) [in press; *Pseudemys*].

[5] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **21**, 223 (1937–1938); Crozier, W. J., and Wolf, E., *Ibid.*, **22** (1938–1939) [in press; *Asellus; Cambarus*].

[6] Crozier, W. J., *Proc. Nat. Acad. Sci.*, **10**, 461 (1924); *Jour. Gen. Physiol.*, **7**, 189 (1924–1925).

[7] Crozier, W. J., and Stier, T. J. B., *Jour. Gen. Physiol.*, **9**, 49 (1925–1926).

[8] Crozier, W. J., *Jour. Gen. Physiol.*, **7**, 189 (1924–1925).

[9] Korr, I. M., *Jour. Cell. Comp. Physiol.*, **10**, 461 (1937).

[10] An analysis of data on this situation will be given elsewhere.

[11] A fuller account is appearing in: Crozier, W. J., and Wolf, E., *Jour. Gen. Physiol.* (1938–1939).

[12] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Ibid.*, **20,** 211 (1936–1937); **21,** 463; etc. (1937–1938).

[13] Crozier, W. J., and Wolf, E., *Ibid.*, **22** (1938–1939) [in press, Sunfish No. 2].

[14] Cf. Pincus, G., *Ibid.*, **14,** 421 (1930–1931).

[15] Wolf, E., and Zerrahn-Wolf, G., *Ibid.*, **19,** 495 (1935–1936); Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., **20,** 211, 411 (1936–1937); **21,** 313 (1937–1938).

[16] Crozier, W. J., and Stier, T. J. B., *Ibid.*, **7,** 699 (1924–1925); Crozier, W. J., *Ibid.*, **9,** 531 (1925–1926).

[17] Crozier, W. J., and Stier, T. J. B., *Ibid.*, **7,** 429 (1924–1925); **9,** 547 (1925–1926); Crozier, W. J., **18,** 801 (1934–1935).

[18] Hecht, S., *Physiol. Rev.*, **17,** 239 (1937).

[19] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **20,** 393 (1936–1937); Hecht, S., The Harvey Lectures (1938), 35 (**1937–38**); *Jour. Appl. Phys.*, **9,** 156.

[20] This can best be tested only in cases not structurally complicated; cf.[4,5] etc.

[21] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Proc. Nat. Acad. Sci.*, **24,** 125, 538 (1938).

[22] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **21,** 17 (1937–1938); *Proc. Nat. Acad. Sci.*, **23,** 516 (1937); Crozier, W. J., and Wolf, E., *Ibid.*, **24,** 542 (1938); *Jour. Gen. Physiol.* [in press] (1938–1939).

[23] Crozier, W. J., *Proc. Nat. Acad. Sci.*, **22,** 412 (1936); Crozier, W. J., and Holway, A. H., *Proc. Nat. Acad. Sci.*, **23,** 23 (1937).

[24] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **21,** 313 (1937–1938).

EXPRESSIONS FOR THE CURRENT IN THE BLOCH APPROXIMATION OF "TIGHT BINDING" FOR METALLIC ELECTRONS

BY J. BARDEEN AND J. H. VAN VLECK

UNIVERSITY OF MINNESOTA AND HARVARD UNIVERSITY

Communicated January 14, 1939

Various approximate methods have been used for the calculation of metallic wave functions.[1] Of these, the most important are: (1) the approximation of "tight binding" in which it is assumed that the atoms are relatively far away from one another, so that the overlap of the wave functions of the electrons in neighboring atoms is small;[2] (2) the opposite limiting case, the approximation of nearly free electrons, in which it is assumed that the variations in the potential energy are small in comparison with the kinetic energy of the metallic electrons[3] and (3) the method of Wigner and Seitz[4] and its extensions by Slater,[5] which is the most accurate of all. We wish to discuss the expressions for the current in the first of these methods. In this method, the Bloch wave functions are approximated by a linear combination of atomic wave functions.

The Bloch form for the wave function of a metallic electron is ψ_k =

$\exp(i\mathbf{k}\cdot\mathbf{r})u_k(\mathbf{r})$, where $u_k(\mathbf{r})$ has the translational period of the crystal lattice. The vector $\mathbf{k}$ is the propagation vector of the electron wave, and is in magnitude $2\pi/\lambda$, where λ is the wave-length. The standard expression for, say, the x-component of the current carried by an electron in this state is:

$$i_x = -ev_x = -(eh/4\pi\mu i)\int[\psi_k^*(\partial\psi_k/\partial x) - \psi_k(\partial\psi_k^*/\partial x)]d\tau. \quad (1)$$

The Bloch expression for the same component of the current is[6]

$$i_x = -(2\pi e/h)(\partial E/\partial \mathbf{k}_x) \quad (2)$$

where E is the energy, expressed as a function of the propagation vector $\mathbf{k}$. These two expressions are, of course, equal if ψ_k is an exact solution of the Schrödinger equation:

$$-(h^2/8\pi^2\mu)\Delta\psi_k + V(\mathbf{r})\psi_k = E_k\psi_k \quad (3)$$

in which $V(\mathbf{r})$ is the periodic potential of the crystal lattice. If ψ_k is determined by some approximate method, there is no reason to suppose that (1) and (2) will give the same result. In fact, if hydrogen-like atomic wave functions, $e^{-\alpha r}$, are used for the calculation of the Bloch wave functions in the approximation of "tight-binding" it is found, as will be shown more explicitly later, that (1) is just one-third of (2).[7]

Such disagreement on the current is perhaps at first surprising, because one might expect that, at least in the limiting case when the distance between the atoms is very large, the approximate wave function so derived will be very close to the exact wave function, obtained by solution of the Schrödinger equation. However, in this limit the current goes to zero, and while both (1) and (2) approach zero, the ratio between them does not approach unity.

It is the purpose of the present paper to point out this discrepancy, explain somewhat its causes and to examine the type of wave function for which exact agreement is to be expected.

Let us first examine the nature of the field in which the electron moves. We may assume that the potential in the crystal lattice is fairly constant in the regions between the atoms, and, without loss of generality, this constant potential may be taken equal to zero. If the potential in the neighborhood of any atom is $U(r)$, the total potential in the crystal is[8]

$$V(\mathbf{r}) = \Sigma_l U(|\mathbf{r} - \rho_l|), \quad (4)$$

where the sum is over all lattice points (designated by the vectors ρ_l). Here $U(r)$ cannot be taken to be the true atomic potential (i.e., the field of the ion about which the valence electron of the free atom moves), as is generally assumed, because the latter will fall off as $1/r$ for large r, and the sum (4) will be infinite for an infinite lattice. Rather, one should use a screened field which in the neighborhood of any atom is very nearly equal

to the field of the ion, but which falls off exponentially instead of as $1/r$ at large distances. The screening of course is due to all the valence electrons except the one under consideration. These considerations indicate that to compute the Bloch functions one should use, instead of the true atomic wave functions, functions which are determined from a screened ionic field. We show below that if this is done the expressions (1) and (2) give the same value for the current, at least in the limiting case when the atoms are far apart. The disagreement on the current mentioned above, which occurs if true atomic wave functions are used, thus seems to be connected with the choice of the crystalline potential.

We proceed with the mathematical proof of the foregoing statements. The equation for the "atomic" wave functions $\varphi(r)$ from which we will construct the Bloch wave function is

$$-(h^2/8\pi^2\mu)\,\Delta\varphi(r) + U(r)\varphi(r) = E_0\varphi(r). \tag{5}$$

For simplicity we assume that $\varphi(r)$ is a radial (s) function. We also assumed that $U(r)$ falls off much more rapidly than $1/r$ for large r, so that the asymptotic form for $\varphi(r)$ which holds if $U(r) \backsimeq 0$ is:

$$\varphi(r) \sim e^{-\alpha r}/r, \tag{6}$$

in which α is determined from

$$E_0 = -\alpha^2h^2/8\pi^2\mu. \tag{7}$$

The true atomic wave function would instead have the asymptotic form $e^{-\alpha r}$, which is appropriate for an electron moving in a coulomb field.

The wave function of an electron moving in the crystal lattice is formed from an appropriate linear combination of the "atomic" wave functions.[8]

$$\psi_k = \Sigma_l \exp[i\mathbf{k}\cdot\rho_l]\ \varphi(|\mathbf{r} - \rho_l|), \tag{8}$$

the sum running over all lattice points.

In this approximation the energy, E_k, is:[8]

$$E_k = E_0 + \Sigma_l \exp[-i\mathbf{k}\cdot\rho_l] \int \varphi(|\mathbf{r} - \rho_l|)\Big(V(\mathbf{r}) - U(r)\Big)\varphi(r)d\tau. \tag{9}$$

The Bloch expression for the velocity of an electron in the state k is

$$v_x = (2\pi/h)(\partial E/\partial k_x) = -(2\pi/h)\Sigma_l\, i(\rho_l)_x I_l \exp[-i\mathbf{k}\cdot\rho_l], \tag{10}$$

in which I_l is the integral:

$$\begin{aligned} I_l &= \int \varphi(|\mathbf{r} - \rho_l|)\Big(V(\mathbf{r}) - u(r)\Big)\varphi(r)d\tau \\ &= \int \varphi(|\mathbf{r} - \rho_l|)\Big(\sum_{m \neq 0} u(|\mathbf{r} - \rho_m|)\Big)\varphi(r)d\tau. \end{aligned} \tag{11}$$

The second form follows from the first by substitution of the expression (4) for $V(\mathbf{r})$. Since we are assuming that $U(|\mathbf{r} - \rho_m|)$ is small except in

the neighborhood of ρ_m, the only significant term in the sum over m is that for which $m = l$, so that

$$I_l \simeq \int \varphi(|\mathbf{r} - \rho_l|)\, U(|\mathbf{r} - \rho_l|)\varphi(r)d\tau. \tag{12}$$

Using the Schrödinger equation (5) for $\varphi(r)$, we see that I_l may be written in the form:

$$I_l \simeq \int \left\{ \left((h^2/8\pi^2\mu)\,\Delta + E_0 \right)\varphi(|\mathbf{r} - \rho_l|) \right\} \varphi(r)d\tau. \tag{13}$$

Since

$$\Delta\varphi(|\mathbf{r} - \rho_l|) = \Delta_\rho \varphi(|\mathbf{r} - \rho_l|), \tag{14}$$

(the second Laplacian is taken with ρ_l as the variable) we may express I_l in terms of the overlap integral

$$g(\rho) = \int \varphi(|\mathbf{r} - \rho|)\varphi(r)d\tau. \tag{15}$$

Thus

$$\begin{aligned} I_l &= \{(h^2/8\pi^2\mu)\,\Delta_\rho + E_0\} g(\rho) \\ &= \left\{ \frac{h^2}{8\pi^2\mu} \frac{d^2g}{d\rho^2} + \frac{2}{\rho}\frac{dg}{d\rho} + E_0 g \right\}_{\rho=\rho_l}. \end{aligned} \tag{16}$$

To obtain the second form we have made use of the fact that $\varphi(r)$ and consequently $g(r)$ are radial functions.

The ordinary expression for the velocity is obtained by substituting the expression (8) for ψ_k into equation (1). We then have:

$$v_x = (ih/2\pi\mu)\Sigma_l \exp[-i\mathbf{k}\cdot\rho_l] \int \varphi(r)(\partial\varphi(|\mathbf{r} - \rho_l|)/\partial x)d\tau. \tag{17}$$

Now

$$\partial\varphi(|\mathbf{r} - \rho_l|)/\partial x = -\partial\varphi(|\mathbf{r} - \rho_l|)/\partial(\rho_l)_x, \tag{18}$$

so that

$$\begin{aligned} v_x &= -(ih/2\pi\mu)\Sigma_l[\partial g(\rho_l)/\partial(\rho_l)_x] \exp[-i\mathbf{k}\cdot\rho_l] \\ &= -(ih/2\pi\mu)\Sigma_l[(\rho_l)_x/\rho_l][dg(\rho_l)/d\rho_l] \exp[-i\mathbf{k}\cdot\rho_l]. \end{aligned} \tag{19}$$

In view of (16), the two expressions (10) and (19) for the velocity will be equal only if

$$\frac{h^2}{8\pi^2\mu}\left[\frac{d^2g}{d\rho^2} + \frac{2}{\rho}\frac{dg}{d\rho}\right] + E_0 g = \frac{h^2}{8\pi^2\mu}\left(\frac{2}{\rho}\frac{dg}{d\rho}\right), \tag{20}$$

or if (cf. Eq. (7))

$$d^2g/d\rho^2 - \alpha^2 g = 0. \tag{21}$$

Thus we must have

$$g = ce^{-\alpha\rho} \tag{22}$$

where c is a constant.

The overlap integral $g(\rho)$ will have this form if the major contribution to the integral comes from the region where the potential is fairly constant (and, by assumption, equal to zero) because in this region the wave function will have the form $e^{-\alpha r}/r$. It is not difficult to show that the overlap integral, $\int \frac{e^{-\alpha|\mathbf{r}-\boldsymbol{\rho}|}}{|\mathbf{r}-\boldsymbol{\rho}|}\frac{e^{-\alpha r}}{r}\,d\tau$, is proportional to $e^{-\alpha\rho}$, and thus has the desired form.

On the other hand, for hydrogen-like wave functions of the form $e^{-\alpha r}$, the overlap integral is proportional to[9]

$$e^{-\alpha\rho}\{1 + \alpha\rho + {}^1/_3(\alpha\rho)^2\},$$

and with tnis expression (19) is just one-third of (10).

[1] See, for example, M. F. Mott and H. Jones, *Theory of the Properties of Metals and Alloys* (Oxford, 1936) Chap. II, or A. Sommerfeld and H. Bethe, *Handbuch der Physik*, 2nd Edition, **24**, 2, p. 370, Berlin (1934).

[2] F. Bloch, *Zeits. Physik*, **52**, 555 (1928).

[3] R. Peierls, *Ann. Physik*, **4**, 121 (1930).

[4] E. Wigner and F. Seitz, *Phys. Rev.*, **43**, 804 (1933).

[5] J. C. Slater, *Ibid.*, **45**, 794 (1934).

[6] Cf. Mott and Jones (reference 1), p. 93.

[7] Analogous considerations apply to the corresponding two ways of computing the diamagnetic current induced by a magnetic field. [F. London, *Jour. de Physique*, **8**, 397 (1937).] If the diamagnetism is computed directly from an integral similar to (1) rather by differentiation with respect to the field strength [the analog of (10)] and if hydrogenic wave functions are used, London's estimate of the resonance integral necessary to account for the diamagnetism of aromatic compounds is raised from 4.4 to 13.2 electron volts. The fact that the latter value is unreasonably high need not cause concern, as the hydrogenic model is not really applicable. Anyway, the change by a factor of three cancels out in computing the relative amounts of diamagnetism of different aromatic substances, so that the ability of his theory to predict the variation of diamagnetism with chemical composition is not affected.

[8] Cf. Mott and Jones (reference 1), p. 65 and following.

[9] Cf. W. Heitler and F. London, *Zeits. Physik*, **44**, 455 (1927).

THE OXIDATION-REDUCTION POTENTIALS OF LAKE WATERS AND THEIR ECOLOGICAL SIGNIFICANCE

BY G. E. HUTCHINSON, E. S. DEEVEY, JR.,[1] AND ANNE WOLLACK

THE OSBORN ZOÖLOGICAL LABORATORY OF YALE UNIVERSITY

Communicated January 3, 1939

Since the classical studies of Thienemann,[2] it has been realized that the nature of the profundal Chironomid fauna of a lake provides a valuable criterion in lake classification. It has usually been believed that oligotrophic lakes with a rich supply of hypolimnetic oxygen have a *Tanytarsus* fauna, eutrophic lakes, with a poor supply, a *Chironomus* fauna. Lundbeck[3] elaborated this generalization, characterizing various mesotrophic categories, without adding much information as to the chemical conditions of the different subtypes. The work of Miyadi[4] indicates that inconsistencies, notably involving "oligotrophic *Chironomus*" lakes, may arise in some regions. In the course of a study of the benthic faunae of Connecticut lakes, one of us (E. S. D.) has recognized comparable inconsistencies, as for instance when Lake Quassapaug, Middlebury, is compared with Job's Pond, Portland. At the end of summer stagnation the oxygen content of the bottom water of the former lake is, in spite of its greater depth, lower than that of the latter, yet Quassapaug is a typical *Tanytarsus* lake, while Job's Pond resembles Miyadi's "oligotrophic *Chironomus*" lakes. Chemical studies (Hutchinson[5]) have indicated large amounts of ferrous iron in the hypolimnion of Linsley Pond, a typical eutrophic *Chironomus* lake, while Jahn[6] found that the bottom water of Quassapaug has a redox potential but little lower than that of the surface water; comparison of Yoshimura's[7] figures for hypolimnetic iron in some of the lakes studied by Miyadi strengthened the suspicion that oxidation-reduction potentials might be of interest in relation to the Chironomid typology of lakes. We have accordingly measured the potentials set up on a bright platinum electrode immersed in freshly collected lake waters, and at the same time have determined ferrous or total iron, total manganese, pH and oxygen. All potentials were determined between 15° and 20°C.; to the nearest centivolt, they may be taken as correct for 18°C. Hill's[8] α-α' dipyridyl method was used in the determination of ferrous iron.

In general the freely circulating epilimnetic waters give potentials of from 0.46–0.50 volt, referred to the standard hydrogen electrode; such values are to be expected in well-oxygenated waters in the absence of reductants (Cooper[9]). Two lakes give somewhat lower surface values (Queechy, Canaan, N. Y., 0.43 volt, and Wononskopomuc, Salisbury, 0.44 volt). In the hypolimnion of most lakes the potential is found to

vary with the amount of ferrous iron present, though owing to the high dilution and complex composition of the water theoretical treatment cannot be attempted. In only one case, that of the bottom water of Queechy, are both the redox potential and ferrous iron (Eh 0.15 volt, Fe^{++} 0.4 mgms. per liter) so low as to suggest the presence of a non-ferrous poising system.

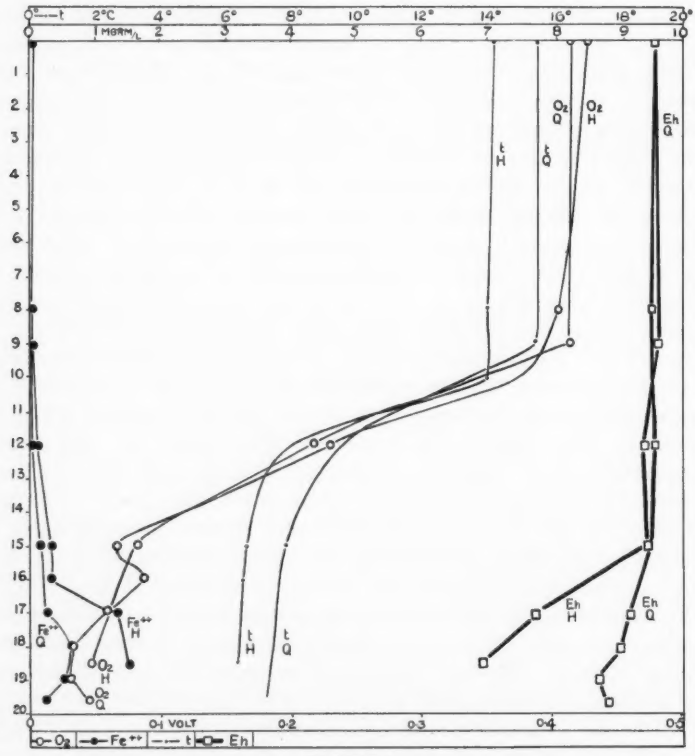

FIGURE 1

Data for Lake Quassapaug (Q), 3 Oct. 1938 and for Highland Lake (H) 7 Oct. 1938.

In figure 1, data for Lake Quassapaug, with an oligotrophic *Tanytarsus* fauna, and for Highland Lake, Winchester, Connecticut, with a mesotrophic *Endochironomus* fauna are set out, while in figure 2 data for the eutrophic and "oligotrophic" *Chironomus* lakes, Linsley Pond, North Branford, Connecticut, and Job's Pond are presented. It will be seen that the Eh curve shows little variation in the *Tanytarsus* lake, slightly greater hypo-

limnetic reduction in the *Endochironomus* lake, a strong microzonal fall in potential in the oligotrophic and marked reduction throughout the hypolimnion in the eutrophic, *Chironomus* lake. All have "eutrophic" oxygen curves. In certain mesotrophic lakes Chironomid larvae are found which, though yielding on emergence adults of a new species of *Chironomus* (s. str.) near *hyperboreus*, have but one pair of, or no, ventral blood gills, a condition common in mesotrophic species. This form is provisionally designated as "mesotrophic *Chironomus*." It occurs with *Tanytarsus* in Lake Wononskopomuc, which shows a very slight fall in potential, to 0.41 volt, at 30.2 m., and without *Tanytarsus* in East Twin

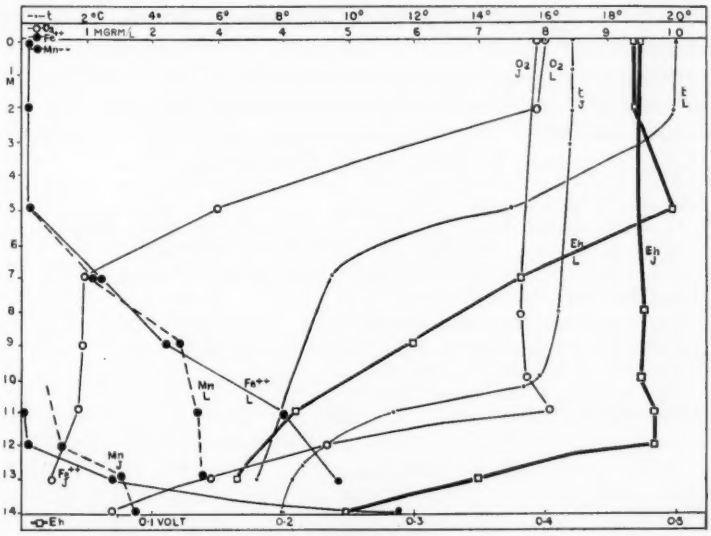

FIGURE 2

Data for Job's Pond (J), 4 Oct. 1938 and for Linsley Pond (L), 16 Sept. 1938.

Lake, Salisbury, where the potential falls microzonally to 0.33 volt at 22.5 m. Mount Tom Pond, a true *Chironomus* lake intermediate in general character between Linsley and Job's Pond, shows a fall from 0.46 volt at 8 m. to 0.26 volt at 13 m. Queechy Lake, a rather oligotrophic true *Chironomus* lake, shows a microzonal fall to 0.15 volt at 0.5 m. above the deepest mud, in this resembling Job's Pond; alone of all the lakes studied a low potential occurred without much ferrous iron being present and a sulphide poising system may well be involved. It would appear therefore that where the water approximately 0.5 m. above the deepest mud has a potential of above 0.4 volt, the mud itself supports a pure or

mixed *Tanytarsus* fauna, where below 0.3 volt a true *Chironomus* (four ventral blood gills) fauna is found, while in the intermediate range either *Endochironomus* or mesotrophic *Chironomus*, with a reduction of ventral blood gills, occurs. It is important to note that muddy water from Wononskopomuc shows that very low potentials (0.15 volt) are prevalent in the mud itself in the vicinity of the animals, so that the relationship here established is primarily of typological interest, indicating the reducing power of the mud as it affects the open water, rather than of significance in the respiratory physiology of the mud-inhabiting animals themselves.

The general data incorporated in the above account were partly obtained from a study supported by the BACHE FUND of the NATIONAL ACADEMY OF SCIENCES and partly from a survey conducted by the Connecticut State Board of Fisheries and Game; the potentiometer used was purchased with a grant from the Sheffield Fund of Yale University. The generosity of these bodies in making our researches possible is here gratefully acknowledged. Our best thanks are also due to Dr. O. A. Johannsen of Cornell University for examining our *Chironomidae*.

[1] Sterling Fellow in Biology.

[2] *Int. Rev. Hydrobiol.*, **6**, 243–249 (1913); *Verh. naturh. Ver. preuss. Rheinl.*, **72**, 1–58 (1917); *Arch. Hydrobiol.*, **12**, 1–65 (1920); *Ibid.*, **13**, 609–646 (1922).

[3] *Arch. Hydrobiol. Suppl.*, **7**, 1–473 (1926); *Ibid.*, **10**, 208–357 (1936).

[4] *Jap. Jour. Zool.*, **4**, 417–437 (1933).

[5] *Proc. Nat. Acad. Sci.*, **24**, 63–69 (1938).

[6] *Cold Spring Harbor Symposia Quant. Biol.*, **2**, 167–180 (1934).

[7] *Jap. Jour. Geol. Geogr.*, **9**, 61–69 (1931).

[8] *Proc. Roy. Soc.*, *B* **107**, 205–214 (1930).

[9] *Jour. Mar. Biol. Assoc.*, **22**, 167–175 (1937).

ON IDEALS OF DIFFERENTIAL POLYNOMIALS

BY J. F. RITT

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated January 4, 1939

Notwithstanding the excellent work of Raudenbush on *perfect* ideals, the general theory of ideals of differential polynomials is in a state which can hardly be called definitive. We hope to increase the plausibility of the existence of a complete theory in proving here a theorem of some generality on *unrestricted* ideals. Applied to the very special case of a single differential equation, in one unknown, of the first order, this theorem provides a substantial idealtheoretic background for the theory of the singular solutions of the equation.

THEOREM: *Let Σ be an ideal of differential polynomials in the unknowns $y_1, \ldots, y_n$. Let the manifold $\mathfrak{M}$ of Σ be composed of manifolds, not necessarily irreducible, $\mathfrak{M}_1, \ldots, \mathfrak{M}_s$, none of which has a solution in common with any other. Then Σ can be represented in one and in only one way as the intersection of s ideals $\Sigma_1, \ldots, \Sigma_s$, the manifold of Σ_i being $\mathfrak{M}_i$.*

One sees that it is sufficient to treat the case of $s = 2$. As $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are mutually exclusive, there exists a relation

$$A + B = 1, \tag{1}$$

where A holds $\mathfrak{M}_1$ and B holds $\mathfrak{M}_2$. For some p, $(AB)^p$ is in Σ. We raise both sides of (1) to the $2p$th power. In the first member of the resulting equation, let M represent the sum of those terms in which the exponent of A exceeds p, and N the sum of the remaining terms. Then $M + N = 1$ and MN is in Σ.

We shall prove that *the derivative M' of M is in Σ*. The derivative of MN, which equals $M'(2N - 1)$, is in Σ. Thus Σ contains $M'(4N^2 - 1)$. Because N^2M' is in Σ, Σ contains M'. N' is also in Σ.

Now let Σ_1 and Σ_2 be, respectively, the ideals generated by $\Sigma + M$ and by $\Sigma + N$. Then Σ_1 and Σ_2 have $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively, as manifolds. We shall prove that Σ *is the intersection of* Σ_1 *and* Σ_2.

It suffices to show that if a form G is contained in Σ_1 and in Σ_2, G is contained in Σ. Such a G, because it is in Σ_1, has, by what precedes, an expression $C + DM$, with C in Σ. As G is in Σ_2, DM is in Σ_2. Because $M = 1 - N$ and N is in Σ_2, D is in Σ_2. Let $D = E + FN$ with E in Σ. Because Σ contains MN, Σ contains DM and hence G.

We settle now the question of uniqueness. Let Σ be the intersection of ideals Σ_1' and Σ_2', whose manifolds are, respectively, $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

Let G be any form in Σ_1'. Some power N^t of N is contained in Σ_2'. Then GN^t is in Σ, hence in Σ_1. Because $N = 1 - M$, G is in Σ_1. Again, let H be any form in Σ_1. For some t, Σ_1' contains M^t and hence HM^t. As $M = 1 - N$ and HN is in Σ, Σ_1' contains H. We have proved that Σ_1 *and* Σ_1' *are identical*. So also, then, are Σ_2 and Σ_2'.

What precedes shows that, for any value of s, each Σ_i is obtained by adjoining a suitable form M_i to Σ and taking the ideal generated by $\Sigma + M_i$. The forms of Σ_i will all be of the type $C + DM_i$ with C in Σ. One sees that, if Σ has a basis for which a single exponent can be used, the same will be true for Σ_i, and the least possible exponent for Σ_i will not exceed that for Σ.

The above work goes over, with no essential additions, to forms in several independent variables.

GENERALIZATIONS TO SPACE OF THE CAUCHY AND MORERA THEOREMS

BY MAXWELL READE AND E. F. BECKENBACH

DEPARTMENT OF MATHEMATICS, RICE INSTITUTE

Communicated December 15, 1938

Introduction.—Consider the function

$$w = f(z) = x_1(u, v) + ix_2(u, v), z = u + iv, \tag{1}$$

defined and continuous in a simply connected domain D. A necessary and sufficient condition that $f(z)$ be analytic in D is that the Cauchy-Riemann equations be satisfied there:

$$\lambda w = 0, \tag{2}$$

where

$$\lambda = \frac{\partial}{\partial u} + i\frac{\partial}{\partial v}$$

is a differential operator. From (2) we obtain

$$\sum_{j=1}^{2} (\lambda x_j)^2 = 0. \tag{3}$$

According to the Cauchy and Morera theorems, a necessary and sufficient condition that the continuous function $f(z)$ in (1) be analytic in the simply connected domain D is that for each circle C lying in D,

$$\int_C f(z)dz = 0. \tag{4}$$

Now (4) may be considered to be an integral analogue of the differential condition (2); it implies

$$\sum_{j=1}^{2} [\int_C x_j(u, v)dz]^2 = 0, \tag{5}$$

which is analogous to (3).

The real functions

$$x_j = x_j(u, v), \qquad j = 1, 2, 3, \tag{6}$$

defined and continuous in a simply connected domain D, will be said to define a surface S. A generalization of (3) to space is

$$\sum_{j=1}^{3} (\lambda x_j)^2 = 0, \tag{7}$$

which characterizes isothermic maps; that is, maps which are conformal except at points where $E = G = 0$.

An analogous generalization of (5) to space is

$$\sum_{j=1}^{3} [\int_C x_j(u, v)dz]^2 = 0. \tag{8}$$

In this note we summarize some of the results which we have obtained in the study of equation (8). A more complete discussion will appear in a subsequent publication.

We shall have frequent use for the following direct computation. Let the functions (6) have continuous partial derivatives of the mth order in a simply connected domain D, so that they admit finite Taylor expansions about each point (u_0, v_0) of D,

$$x_j(u, v) = \sum_{n=0}^{m} \frac{r^n}{n!}\left[\left(\cos\theta \frac{\partial}{\partial u} + \sin\theta \frac{\partial}{\partial v}\right)^n x_j\right] + \sigma(r^m), j = 1, 2, 3, \tag{9}$$

where the partial derivatives are evaluated at (u_0, v_0), where $\sigma(r^\alpha)$ denotes a quantity (not always the same quantity) such that

$$\lim_{r \to 0} \frac{\sigma(r^\alpha)}{r^\alpha} = 0,$$

and where

$$u - u_0 = r\cos\theta, v - v_0 = r\sin\theta.$$

Then, for the circle C with center (u_0, v_0) and radius r in D, the left-hand member of (8) assumes the form

$$\sum_{j=1}^{3} [\int_C x_j(u, v)dz]^2 = -\pi^2 \sum_{j=1}^{3} \sum_{k=1}^{m} \sum_{s=1}^{k} \frac{r^{2k+2}}{2^{2k-2}(k!)^2} {}^kC_s {}^kC_{s-1} (\Delta^{s-1}\lambda x_j)(\Delta^{k-s}\lambda x_j) + \sigma(r^{2m+2}), \tag{10}$$

where

$$\Delta \equiv \lambda\bar{\lambda} \equiv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \equiv \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2},$$

and where the ${}^kC_s$ are binomial coefficients,

$${}^kC_s \equiv \frac{k!}{s!(k-s)!}.$$

I. *Characterization of Isothermic Maps.*—THEOREM 1. *If the functions* (6) *have continuous partial derivatives of the first order in a simply connected domain* D, *then a necessary and sufficient condition that they map* D *isothermically on a surface* S *is that for each point* (u_0, v_0) *of* D,

$$\sum_{j=1}^{3} [\int_C x_j(u, v)dz]^2 = \sigma(r^4), \tag{11}$$

where C *is the circle with center at* (u_0, v_0) *and radius* r *in* D.

PROOF: With $m = 1$ in (9), (10) becomes

$$\sum_{j=1}^{3} \int_C [x_j(u, v)dz]^2 = -\pi^2 r^4 \sum_{j=1}^{3} (\lambda x_j)^2 + \sigma(r^4). \tag{12}$$

From (12), it follows that (7) is both necessary and sufficient for (11).

II. *Characterization of Those Isothermic Spherical Maps Which Do Not Map Circles on Circles.*—LEMMA. *If the functions* (6) *are not identically constant, and if they map* D *isothermically on a surface* S *that lies on a sphere* Q *of finite radius, then a necessary and sufficient condition that they map circles on circles is that the first quadratic form of* S *have the expression*

$$ds^2 = \frac{c^2(du^2 + dv^2)}{[(u - u_0)^2 + (v - v_0)^2 + a^2]^2}, \; a > 0, \; c > 0. \tag{13}$$

NECESSITY. Consider the stereographic projection of Q on the (x_1, x_2)-plane, and let the map of S be D'. The product of the transformation (6) and the stereographic projection maps D on D', and carries circles in D into circles in D'. It follows that the product of the two transformations is equivalent to a single linear transformation, whence a computation yields ds^2 in the form (13).

SUFFICIENCY. Let the stereographic projection of D on the sphere Q' be S', where Q' is the sphere with center at $(u_0, v_0, a - c/2a)$ and radius $\rho = c/2a$, and where the pole of projection is at (u_0, v_0, a). The first quadratic form of S' is found to be identical with that of S, so that S and S' are applicable. Therefore S and S' are either congruent or symmetric; in either case, since circles in D are mapped on circles on S', it follows that circles in D are mapped on circles on S.

We note that if one non-null circle in D is mapped on a circle on the spherical surface S by the isothermic functions (6), then all circles in D are mapped on circles on S.

THEOREM 2. *If the functions* (6) *have continuous partial derivatives of the third order in a simply connected domain* D, *then a necessary and sufficient condition that they map* D *isothermically on a surface* S *that lies on a sphere of finite radius, such that circles are not mapped on circles, is that*

$$\sum_{j=1}^{3} [\int_C x_j(u, v)dz]^2 = \sigma(r^\alpha) \tag{14}$$

hold identically in D *for* $\alpha = 6$, *but not for* $\alpha = 8$.

NECESSITY. With $m = 3$ in (9), (10) becomes

$$\sum_{j=1}^{3} [\int_C x_j(u, v)dz]^2 = -\pi^2 \sum_{j=1}^{3} \left\{ r^4(\lambda x_j)^2 + \frac{r^6}{4} (\lambda x_j)(\Delta\lambda x_j) + \frac{r^8}{192} [2(\lambda x_j)(\Delta^2\lambda x_j) + 3(\Delta\lambda x_j)^2] \right\} + \sigma(r^8). \quad (15)$$

For our isothermic spherical map, we have

$$E = G,\ F = 0;\ e = g,\ f = 0;\ e = kE;\ k = \text{const.} \neq 0, \quad (16)$$

where E, F, G and e, f, g are the coefficients of the first and second fundamental quadratic forms for S. The formulas of Gauss, simplified by (16), yield

$$\sum_{j=1}^{3} (\lambda x_j)(\Delta\lambda x_j) = 0. \quad (17)$$

It follows from (7), (17) and (15) that (14) holds for $\alpha = 6$.

Suppose that (14) holds identically in D for $\alpha = 8$. Then, by (15), we have (7), (17) and

$$2\sum_{j=1}^{3} (\lambda x_j)(\Delta^2\lambda x_j) + 3\sum_{j=1}^{3} (\Delta\lambda x_j)^2 = 0.$$

A computation, using (16), yields

$$\sum_{j=1}^{3} (\lambda x_j)(\Delta^2\lambda x_j) = -4k^2E(\lambda^2 E), \quad \sum_{j=1}^{3} (\Delta\lambda x_j)^2 = 4k^2(\lambda E)^2,$$

whence

$$3(\lambda E)^2 - 2E(\lambda^2 E) = 0. \quad (18)$$

It follows that

$$\lambda E = E^{3/2}e^{\varphi(z)}, \quad (19)$$

where $\lambda\varphi(z) = 0$, so that $\varphi(z)$ is analytic in D. From the imaginary part of the left-hand member of (18), we obtain

$$\frac{3E_u du}{E} = \frac{2E_{uv}du}{E_v}, \quad \frac{3E_v dv}{E} = \frac{2E_{uv}dv}{E_u},$$

whence

$$\lambda E = E^{3/2}(e^{\varphi_1(u)} + ie^{\varphi_2(v)}). \quad (20)$$

It follows from (19), (20) and the Cauchy-Riemann equations, that

$$e^{\varphi_1(u)} = a_0u + a_1,\ e^{\varphi_2(v)} = a_0v + a_2, \quad (21)$$

where a_0, a_1, a_2 are real constants. Substituting from (21) in (20), and integrating, we find

$$E = \frac{4}{\left[\frac{a_0}{2}(u^2 + v^2) + a_1u + a_2v + a_3\right]^2}. \tag{22}$$

Since on S the Gaussian curvature K is positive, it follows from

$$K = -\frac{1}{2E}\,\Delta \log E$$

that (22) may be written in the form (13). Hence, by the above lemma circles in D are mapped on circles on S, contrary to hypothesis.

SUFFICIENCY. Applying the differential operator $\bar{\lambda}$ to (7), we obtain

$$\sum_{j=1}^{3} (\lambda x_j)(\Delta x_j) = 0, \tag{23}$$

and then applying λ to (23), and using (17), we obtain

$$\sum_{j=1}^{3} (\lambda^2 x_j)(\Delta x_j) = 0. \tag{24}$$

The four real linear homogeneous equations in Δx_j, $j = 1, 2, 3$, implied by (23) and (24), have a solution other than $\Delta x_j = 0$, $j = 1, 2, 3$, only if the rank t of the matrix of coefficients of the system is less than three; but a computation shows that $t < 3$ implies $e = g$, $f = 0$. Hence, (7) and (17) imply that D is mapped isothermically on either a minimal surface or a sphere.

If D is mapped isothermically on a minimal surface, that is, if (7) and $\Delta x_j = 0$, $j = 1, 2, 3$, are satisfied, then, by (10), (8) is satisfied for each circle C in D; in particular, (14) is satisfied identically in D for $\alpha = 8$, contrary to hypothesis. Or if D is mapped isothermically on a surface S that lies on a sphere Q, and if circles in D are mapped on circles on S, we consider any circle C in D and its image C' on S, and project Q stereographically on the plane Π of C', with D' being the map of D on Π; there is induced an isothermic mapping

$$x_j = y_j(u, v), \quad j = 1, 2, 3,$$

of D on D', that is, on a minimal surface; hence,

$$\sum_{j=1}^{3} [\textstyle\int_C y_j(u, v)dz]^2 = 0;$$

but on C,

$$x_j(u, v) \equiv y_j(u, v), \quad j = 1, 2, 3,$$

so that again (8) holds.

Hence, under our hypotheses, D is mapped isothermically on a spherical surface of finite radius, and circles are not mapped on circles.

III. *Characterization of Those Isothermic Spherical Maps Which Map Circles on Circles and of Isothermic Maps on Minimal Surfaces.*—THEOREM 3. *If the functions* (6) *have continuous partial derivatives of the third order in a simply connected domain* D, *then a necessary and sufficient condition that they map* D *isothermically either on a spherical surface, carrying circles into circles, or on a minimal surface, is that* (8) *hold for each circle* C *in* D; *further* (14), *with* $\alpha = 8$, *is equivalent to* (8).

The proof is contained in the proof of Theorem 2.

AN EQUIVALENCE THEOREM FOR SERIES OF ORTHOGONAL POLYNOMIALS

BY G. H. PEEBLES

DEPARTMENT OF MATHEMATICS AND MECHANICS, INSTITUTE OF TECHNOLOGY, UNIVERSITY OF MINNESOTA

Communicated January 9, 1939

1. *Introduction.*—A set of polynomials $[p_n(x) = c_n x^n + \ldots]$ which satisfy the relations

$$\int_a^b \rho(x) p_n(x) p_m(x) dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases} \tag{1}$$

where $\rho(x)$ is non-negative on (a, b) and such that $\int_a^b \rho(x)dx > 0$, may be used to expand formally an "arbitrary" function $f(x)$:

$$f(x) \sim \sum_{n=0}^{\infty} p_n(x) \int_a^b \rho(t) f(t) p_n(t) dt = \sum_{n=0}^{\infty} b_n p_n(x). \tag{2}$$

One may then consider the problem of showing for what interval, if any,

$$\lim_{n \to \infty} \sum_{k=0}^{n} b_k p_k(x) = f(x).$$

Concerning one of the several methods of attacking this problem, Shohat says,[1] "The supreme goal, when dealing with [(2)], is to show that, regarding convergence, it behaves in a certain subinterval of (a, b) like the ordinary Fourier series expansion of $f(x)$ or of some function simply related to it. This constitutes what we call the equiconvergence theorem,

of greatest importance in our theory. Indeed, we well know the wide range of validity of the Fourier series expansion, which makes it such a powerful analytical instrument. The classes of [orthogonal polynomials] for which this goal is attainable are evidently the most interesting, and the method by which the goal has been attained is evidently the most powerful one."

This goal has been attained by A. Haar[2] and W. H. Young[3] for $\rho(x) = 1$ (Legendre polynomials) and by G. Szegö[4] for a more general weight function. The purpose of this paper is to present a compact and self-contained proof of equivalence, without the use of asymptotic formulas, which by reason of its simplicity seems to render the whole theory much more accessible, and at the same time, without reproducing all the results of more elaborate investigations, leads incidentally to an extensive range of conclusions which are believed to be new.

A polynomial expansion which is shown to be equivalent to a Legendre polynomial expansion is in turn equivalent to a Fourier series expansion, if the restrictions given by Haar and Young hold. A more direct connection, however, can be made with a Fourier series expansion by means of the so-called trigonometric polynomials. If the interval (a, b) is taken for simplicity as $(-1, 1)$, the trigonometric polynomials are those orthogonal with respect to the weight function $(1 - x^2)^{-1/2}$. Relation (2) for these polynomials becomes by the substitution $x = \cos\theta$ the Fourier series expansion of the even function $f(\cos\theta)$.

The method used in proving the equivalence theorem will be seen to be applicable to systems of orthogonal functions other than polynomials.[5] It depends on a simple relation for the difference of two partial sums. Suppose the system of polynomials $[p_n(x;\ \rho_1)]$ are orthogonal and normalized with respect to the weight function $\rho_1(x)$, the polynomials $[p_n(x;\ \rho_2)]$ are orthogonal and normalized with respect to the weight function $\rho_2(x)$, where, here and hereafter, it is understood that $\rho_1(x)$ and $\rho_2(x)$ are non-negative, positive on a set of positive measure and of class $L(a, b)$. Let $s_n(x;\ \rho_1) = \sum_{i=0}^{n} b_i p_i(x;\ \rho_1)$ and $s_n(x;\ \rho_2) = \sum_{i=0}^{n} a_i p_i(x;\ \rho_2)$ be the nth partial sums of the formal expansions in terms of the two systems of a function $f(x)$ of classes $L\rho_1(a, b)$, $L^2\rho_1(a, b)$ and $L\rho_2(a, b)$. The difference $s_n(x;\ \rho_2) - s_n(x;\ \rho_1)$ is a polynomial of degree n at most; so

$$s_n(x;\ \rho_2) - s_n(x;\ \rho_1) = \sum_{k=0}^{n} r_{kn} p_k(x;\ \rho_2), \tag{3}$$

where r_{kn} depends on n and is given by the formula

$$r_{kn} = \int_a^b \rho_2(x) \sum_{i=0}^{n} [a_i p_i(x;\ \rho_2) - b_i p_i(x;\ \rho_1)]\ p_k(x;\ \rho_2) dx.$$

By virtue of (1), $r_{kn} = a_k - \sum_{i=0}^{n} b_i \int_a^b \rho_2(x)p_i(x;\ \rho_1)p_k(x;\ \rho_2)dx$. Assume that a function $\rho(x)$ of class $L^2\rho_1(a, b)$ exists such that $\rho(x)\rho_1(x) = \rho_2(x)$ almost everywhere in (a, b). Then $\int_a^b \rho_2(x)p_i(x;\ \rho_i)p_k(x;\ \rho_2)dx = \int_a^b \rho_1(x)[\rho(x)p_k(x;\ \rho_2)]p_i(x;\ \rho_1)dx$ and, if B_{ik} represents the last integral, one has from Parseval's formula $\sum_{i=0}^{\infty} b_iB_{ik} = \int_a^b \rho_1(x)\rho(x)p_k(x;\ \rho_2)f(x)dx = a_k$. Hence

$$r_{kn} = \sum_{i=n+1}^{\infty} b_iB_{ik}$$

and $\lim_{n\to\infty} r_{kn} = 0$.

Equation (3) with the relation just obtained enables one to place on $\rho(x)$ additional conditions sufficient for $\lim_{n\to\infty} [s_n(x;\ \rho_2) - s_n(x;\ \rho_1)] = 0$ wherever the polynomials $[p_n(x;\ \rho_2)]$ are bounded. It follows that if one series converges to a value, diverges, is summable, the other must likewise converge to the same value, diverge, be summable; if one series is uniformly convergent in an interval where the polynomials $[p_n(x;\ \rho_2)]$ are uniformly bounded, the other is uniformly convergent in this interval. The powerful convergence tests for Fourier series expansions become, through the medium of the function $f(\cos\theta)$, instruments for determining the convergence of any polynomial expansion of $f(x)$ which can be shown to be equivalent to the trigonometric polynomial expansion. The present treatment is believed to constitute, in particular, a valuable simplification in the theory of convergence and summability of Legendre series as well as of more general series of Jacobi polynomials.

2. *Notation.*—Subsequent discussion will be simplified if certain abridgments are permitted. The symbol $\pi(x;\ \lambda)$ will be understood to represent a function of the form $\left|\frac{x - r_1}{b - a}\right|^{\lambda_1} \left|\frac{x - r_2}{b - a}\right|^{\lambda_2} \cdots \left|\frac{x - r_\nu}{b - a}\right|^{\lambda_\nu}$ where always $\lambda_1, \lambda_2, \ldots, \lambda_\nu \geq 0$ and $a \leq r_1 < r_2 < \ldots < r_\nu \leq b$, and the symbol $\pi_m(x;\ \mu)$, a polynomial $\left(\frac{x - r_1}{b - a}\right)^{\mu_1} \left(\frac{x - r_2}{b - a}\right)^{\mu_2} \cdots \left(\frac{x - r_\nu}{b - a}\right)^{\mu_\nu}$ of degree $m = \mu_1 + \mu_2 + \ldots + \mu_\nu$, non-negative on (a, b). The r's are arbitrary points of the interval (a, b), but, in any one instance, if there is occasion to write several such functions, say, $\pi(x;\ \lambda)$, $\pi(x;\ \lambda')$, $\pi_m(x;\ \mu)$, the r's are to be the same for each function. It will be convenient, when setting down conditions preparatory to establishing some contemplated result, to regard $\pi(x;\ \lambda)$ as consisting of a single factor $\left|\frac{x - r}{b - a}\right|^{\lambda}$. This has the advantage of per-

mitting one to replace a set of relations such as $\lambda_1 > \lambda_1'$, $\lambda_2 > \lambda_2'$, ..., $\lambda_\nu > \lambda_\nu'$ by $\lambda > \lambda'$ and, it will be seen, amounts to first proving the result for the simple function and then extending it to the more general function. Products such as $\pi(x; \lambda)\pi(x; \lambda')$ will be written in the form $\pi(x; \lambda + \lambda')$.

Further simplification is effected by substituting the statement, "$\rho_2(x)$ is equivalent to $\rho_1(x)$," for the statement, "the two expansions of $f(x)$ in terms of the polynomials corresponding to the weight functions $\rho_2(x)$ and $\rho_1(x)$ are equivalent."

3. *Preliminary Formulation.*—A special form of relation (3) is required for the procedure given here. Let $\bar{\rho}_2(x) = \pi(x; \lambda)\rho_2(x)$ and without change in the restrictions on and definitions of $\rho_1(x)$, $\rho_2(x)$, $\rho(x)$ and $f(x)$, but with slight modifications of the definitions of r_{kn} and B_{ik}, write relation (3) for $\rho_2(x)$ and $\rho_1(x)$:

$$s_n(x; \bar{\rho}_2) - s_n(x; \rho_1) = \sum_{k=0}^{n} r_{kn} p_k(x; \bar{\rho}_2),$$

$$r_{kn} = \sum_{i=n+1}^{\infty} b_i \int_a^b \rho_1(x)[\pi(x; \lambda)\rho(x)p_k(x; \bar{\rho}_2)]p_i(x; \rho_1)dx = \sum_{i=n+1}^{\infty} b_i B_{ik}.$$

One sees from the definition of B_{ik} that

$$\pi(x; \lambda)\rho(x)p_k(x; \bar{\rho}_2) \sim \sum_{i=0}^{\infty} B_{ik} p_i(x; \rho_1). \tag{4}$$

The existence of $\int_a^b \rho_1(x)[\rho(x)]^2 dx$ assures the existence of

$$\int_a^b \rho_1(x)[\pi(x;\lambda)\rho(x)p_k(x; \bar{\rho}_2)]^2 dx.$$

Hence

$$\sum_{i=0}^{\infty} B_{ik}^2 = \int_a^b \rho_1(x)[\pi(x; \lambda)\rho(x)p_k(x; \bar{\rho}_2)]^2 dx = \int_a^b \pi(x; \lambda)\rho(x)\bar{\rho}_2(x)[p_k(x; \bar{\rho}_2)]^2 dx \tag{5}$$

and

$$\sum_{i=n+1}^{\infty} B_{ik}^2 = \int_a^b \rho_1(x)[\pi(x; \lambda)\rho(x)p_k(x; \bar{\rho}_2) - \sigma_n(x; k)]^2 dx, \tag{6}$$

where $\sigma_n(x; k) = \sum_{i=0}^{n} B_{ik} p_i(x; \rho_1)$.

Let $\pi_m(x; \mu)$ be a polynomial of the kind described above, $\mu \leq \lambda$; and, for every value of $k \leq n - m$, let $P_{n-k-m}(x)$ be a polynomial of degree $n - k - m$ at most. Since the integral in equation (6) is increased, if $\sigma_n(x; k)$ is replaced by any other polynomial, one has, for $k \leq n - m$,

$$\sum_{i=n+1}^{\infty} B_{ik}^2 \leq \int_a^b \rho_1(x)[\pi(x;\lambda)\rho(x)p_k(x;\bar{\rho}_2) - \pi_m(x;\mu)p_k(x;\bar{\rho}_2)P_{n-k-m}(x)]^2dx$$

$$= \int_a^b \rho_1(x)[\pi_m(x;\mu)p_k(x;\bar{\rho}_2)]^2[\pi(x;\lambda-\mu)\rho(x) - P_{n-k-m}(x)]^2dx. \quad (7)$$

Suppose $\pi_m(x;\mu)$ and the P's can be taken so that $[\pi_m(x;\mu)]^2\rho_1(x) \leq \pi(x;\lambda)\rho_2(x)$ and $|\pi(x;\lambda-\mu)\rho(x) - P_{n-k-m}(x)| \leq e_{n-k-m}$, a constant. Then

$$\sum_{i=n+1}^{\infty} B_{ik}^2 \leq e_{n-k-m}^2 \int_a^b \rho_1(x)[\pi_m(x;\mu)]^2[p_k(x;\bar{\rho}_2)]^2dx$$

$$\leq e_{n-k-m}^2 \int_a^b \pi(x;\lambda)\rho_2(x)[p_k(x;\bar{\rho}_2)]^2dx = e_{n-k-m}^2.$$

By Cauchy's inequality,

$$\left|r_{kn}\right| \leq \left\{\sum_{i=n+1}^{\infty} b_i^2\right\}^{1/2}\left\{\sum_{i=n+1}^{\infty} B_{ik}^2\right\}^{1/2} \leq \left\{\sum_{i=n+1}^{\infty} b_i^2\right\}^{1/2} e_{n-k-m}. \quad (8)$$

Hence, if $M(x) \geq |p_k(x;\bar{\rho}_2)|$ in (a, b) for $k = 0, 1, \ldots$, one has from (3)

$$\left|s_n(x;\bar{\rho}_2) - s_n(x;\rho_1)\right| \leq M(x)\left\{\sum_{i=n+1}^{\infty} b_i^2\right\}^{1/2}$$

$$\left\{\sum_{k=0}^{n-m} e_{n-k-m} + \sum_{k=n-m+1}^{n}\left[\sum_{i=n+1}^{\infty} B_{ik}^2\right]^{1/2}\right\}. \quad (9)$$

There is some latitude in the choice of assumptions sufficient to cause the second member of the last inequality to tend to zero as n increases indefinitely. Since more satisfactory results are obtained, in general, if $f(x)$ is lightly restricted, $f(x)$ will be left, as heretofore, a function of classes $L\rho_1(a, b)$ and $L^2\rho_1(a, b)$ and the severest restrictions placed elsewhere.

Suppose $d/dx[\pi(x;\lambda-\mu)\rho(x)]$ satisfies a Lipschitz condition of order α: $|d/dx[\pi(x;\lambda-\mu)\rho(x)]_{x=x_2} - d/dx[\pi(x;\lambda-\mu)\rho(x)]_{x=x_1}| \leq \gamma|x_2 - x_1|^{\alpha}$, γ, a constant, $0 < \alpha \leq 1$. Then, for $n > 1$, there exists a polynomial $P_n(x)$ of degree n at most such that $|\pi(x;\lambda-\mu)\rho(x) - P_n(x)| \leq K/n^{1+\alpha}$ in the interval (a, b), where K depends only on γ and the length of the interval.[6] Hence, the polynomials governing the magnitude of the e's can be so chosen that $\sum_{k=0}^{n-m} e_{n-k-m} \leq \frac{K}{(n-m)^{1+\alpha}} + \frac{K}{(n-m-1)^{1+\alpha}} + \ldots + \frac{K}{2^{1+\alpha}} + e_1 + e_0 <$ a constant for all values of n. Equation (5) shows that $\sum_{i=n+1}^{\infty} B_{ik}^2$ is bounded for all values of n and k, if $\pi(x;\lambda)\rho(x)$ is bounded in (a, b). The number of such terms appearing in (9) is a fixed number m. The assump-

tion that $f(x)$ is of class $L^2\rho_1(a, b)$ implies that $\lim_{n\to\infty} \sum_{i=n+1}^{\infty} b_i^2 = 0$. Therefore, if $M(x_0)$ exists, $\lim_{n\to\infty} [s_n(x_0; \bar{\rho}_2) - s_n(x_0; \rho_1)] = 0$.

As the boundedness of $M(x)$ is important in this result, it is useful to know that the polynomials $[p_n(x; \bar{\rho}_2)]$ are bounded at a value of x which is not a zero of $\pi(x; \lambda)\rho(x)$ and for which the polynomials $[p_n(x; \rho_1)]$ are bounded.[7] The conditions given are sufficient to permit one to replace the sign of formal expansion in (4) by the sign of equality.[8] Since $B_{ik} = \int_a^b \pi(x; \lambda)\rho_2(x)p_k(x; \bar{\rho}_2)p_i(x; \rho_1)dx = 0$, when $i < k$, $\pi(x; \lambda)\rho(x)p_k(x; \bar{\rho}_2) = \sum_{i=k}^{\infty} B_{ik}p_i(x; \rho_1)$. But $|B_{ik}| \leq \left\{\sum_{j=i}^{\infty} B_{jk}^2\right\}^{1/2} \leq \frac{K}{(i-k-m-1)^{1+\alpha}}$, when $i - k - m - 1 > 1$. Hence, if $N(x) \geq |p_k(x; \rho_1)|$ for $k = 0, 1, \ldots$, $\pi(x; \lambda)\rho(x)\left|p_k(x; \bar{\rho}_2)\right| \leq N(x)\left\{\sum_{i=k}^{k+m+2}\left|B_{ik}\right| + \frac{K}{2^{1+\alpha}} + \frac{K}{3^{1+\alpha}} + \ldots + \frac{K}{(n-k-1)^{1+\alpha}} + \ldots\right\}$. As B_{ik} is uniformly bounded, the desired result follows immediately from this inequality.

The conditions which are sufficient for these results are collected for future reference in a

LEMMA. *If* f(x) *is of classes* $L\rho_1(a, b)$ *and* $L^2\rho_1(a, b)$ *and if there is a function* $\rho(x)$ *and a polynomial* $\pi_m(x; \mu)$, $\mu \leq \lambda$, *such that in* (a, b) $\rho_2(x) = \rho(x)\rho_1(x)$ *almost everywhere,* $\pi(x; \lambda)\rho_2(x) \geq [\pi_m(x; \mu)]^2\rho_1(x)$ *and* $d/dx[\pi(x; \lambda - \mu)\rho(x)]$ *satisfies a Lipschitz condition of order* α, *then the two expansions of* f(x) *in terms of the polynomials* $[p_n(x; \bar{\rho}_2)]$ *and* $[p_n(x; \rho_1)]$ *are equivalent wherever the former system is bounded. The polynomials* $[p_n(x; \bar{\rho}_2)]$ *are bounded wherever the polynomials* $[p_n(x; \rho_1)]$ *are bounded except for the zeros of* $\pi(x; \lambda)\rho(x)$.

4. *Equivalence Theorem.*—From the preceding Lemma it is possible to draw a succession of results which can be combined in a single theorem. The first result, however, comes more easily from the procedure leading to the Lemma. Let $\pi(x; \lambda) = \pi_m(x; \mu) = 1$, and let $\rho(x)$ be a polynomial of degree r, non-negative in (a, b). Then for $n - k \geq r$ one can take $P_{n-k}(x) = \rho(x)$ and so have from (7) $\sum_{i=n+1}^{\infty} B_{ik}^2 = 0$. The first inequality of (8) shows, therefore, that $r_{kn} = 0$, $k = 0, 1, \ldots, n - r$, and, since $\sum_{i=n+1}^{\infty} B_{ik}^2$ is bounded for all values of n and k, that r_{kn}, $k = n - r + 1, \ldots, n$, tends uniformly to zero as n increases indefinitely. One has from (3) $\rho(x)\rho_1(x)$ equivalent to $\rho_1(x)$ wherever the polynomials $[p_n(x; \rho\rho_1)]$ are bounded. One also has the polynomials $[p_n(x; \rho\rho_1)]$ bounded where the polynomials $[p_n(x; \rho_1)]$ are bounded except for the roots of $\rho(x)$.[9] Hence, $\rho_1(x)$ and $\rho(x)\rho_1(x)$ are equivalent wherever the polynomials $[p_n(x; \rho_1)]$ are bounded and $\rho(x) \neq 0$.

As a first application of the Lemma, let $\rho_2(x) = \rho_1(x) = \pi(x; \lambda')\bar{\rho}_1(x)$, where $\bar{\rho}_1(x)$ has the properties hitherto attributed to $\rho_1(x)$. In abridged notation, let the integer $\mu \geq \lambda' + 1$ and $\lambda = 2\mu - \lambda'$. Then $\pi(x; \lambda)\rho_2(x) = \pi(x; \lambda)\pi(x; \lambda')\bar{\rho}_1(x) = \pi(x; \lambda + \lambda')\bar{\rho}_1(x) = \pi(x; 2\mu)\bar{\rho}_1(x) \geq \pi(x; 2\mu + \lambda')\bar{\rho}_1(x) = [\pi_m(x; \mu)]^2\rho_1(x)$. Since $\lambda - \mu = \mu - \lambda' \geq 1$ and $\rho(x) = 1$, $d/dx\,[\pi(x; \lambda - \mu)\rho(x)]$ satisfies a Lipschitz condition of order α in (a, b). Hence, by the Lemma, $\bar{\rho}_2(x)$ is equivalent to $\rho_1(x)$ wherever the polynomials $[p_n(x; \bar{\rho}_2)]$ are bounded. But $\bar{\rho}_2(x) = [\pi_m(x; \mu)]^2\bar{\rho}_1(x)$ is also equivalent to $\bar{\rho}_1(x)$ wherever the polynomials $[p_n(x; \bar{\rho}_2)]$ are bounded. Therefore $\pi(x; \lambda')\bar{\rho}_1(x)$ and $\bar{\rho}_1(x)$ are equivalent wherever the polynomials $[p_n(x; \bar{\rho}_2)]$ are bounded, or, using part of the last result, wherever the polynomials $[p_n(x; \bar{\rho}_1)]$ are bounded with the exception of the points $r_1, r_2, \ldots r_\nu$.

To advance still further suppose $\rho_2(x) \geq v\pi(x; \lambda')\rho_1(x)$, v, a positive constant. In the Lemma let λ be an even integer greater than or equal to λ' and let $\mu = \lambda$. Then $\pi(x; \lambda - \mu)\rho(x) = \rho(x)$ and $\bar{\rho}_2(x) = \pi(x; \lambda)\rho_2(x) \geq v\pi(x; \lambda + \lambda')\rho_1(x) \geq v\pi(x; 2\lambda)\rho_1(x) = v[\pi_m(x; \mu)]^2\rho_1(x)$. The Lemma calls for $\bar{\rho}_2(x) \geq [\pi_m(x; \mu)]^2\rho_1(x)$, whereas, here $\bar{\rho}_2(x) \geq v[\pi_m(x; \mu)]^2\rho_1(x)$. But this is readily seen to present no difficulty. Therefore, if $\rho'(x)$ is assumed to satisfy a Lipschitz condition of order α, $\bar{\rho}_2(x)$ and $\rho_1(x)$ are equivalent wherever the polynomials $[p_n(x; \bar{\rho}_2)]$ are bounded. By the preceding result, the same statement can be made for $\bar{\rho}_2(x)$ and $\rho_2(x)$. So $\rho_2(x)$ and $\rho_1(x)$ are equivalent where the polynomials $[p_n(x; \rho_1)]$ are bounded and $\pi(x; \lambda')\rho(x) \not\equiv 0$.

The last case which includes most of the earlier results is contained in the

THEOREM. *If the polynomials* $[p_n(x; \rho_2)]$ *and* $[p_n(x; \rho_1)]$ *are orthogonal and normalized with respect to the weight functions* $\rho_2(x)$ *and* $\rho_1(x)$ *over the interval* (a, b), *if* $f(x)$ *is of classes* $L\rho_1(a, b)$ *and* $L^2\rho_1(a, b)$, *and if a function* $\rho(x)$ *and a set of points* $r_1, r_2, \ldots, r_\nu$, *exist such that in* (a, b) $\rho_2(x) \geq v\prod_{i=1}^{\nu}|x - r_i|^{\lambda_i}\rho_1(x)$, $\rho_2(x) = \rho(x)\rho_1(x)$ *almost everywhere, and* $\rho'(x)$ *satisfies a Lipschitz condition of order* α, *where* v *is a positive constant and the* λ*'s are non-negative constants, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} [p_k(x; \rho_2) \int_a^b \rho_2(t)f(t)p_k(t; \rho_2)dt - p_k(x; \rho_1) \int_a^b \rho_1(t)f(t)p_k(t; \rho_1)dt] = 0$$

wherever the polynomials $[p_n(x; \rho_1)]$ *are bounded and* $\rho(x)\prod_{i=1}^{\nu}|x - r_i|^{\lambda_i} \not\equiv 0$.

This Theorem can be further generalized. If $\rho(x) = (1 - x^2)^{-1/2}$, $\rho'(x)$ does not satisfy a Lipschitz condition. But the derivative of the product $(1 - x^2)^{1/2}\rho(x)$, after removing discontinuities, does. One has, then, $(1 - x^2)^{1/2}\rho_2(x)$ equivalent to $\rho_1(x)$, $(1 - x^2)^{1/2}\rho_2(x)$ equivalent to

$\rho_2(x)$ and, for a function of class $L\rho_2(a, b)$, $\rho_2(x)$ equivalent to $\rho_1(x)$ wherever the polynomials $[p_n(x; \rho_1)]$ are bounded except at -1 and 1. It is clear from this example that one may supplement the Theorem by the statement: *The restriction on* $\rho'(x)$ *may be replaced by two conditions,* $f(x)$ *is of class* $L\rho_2(a, b)$ *and a set of points* $r_1', r_2', \ldots, r_{\nu'}'$ *exists such that in* (a, b) d/dx $[\rho(x) \prod_{i=1}^{\nu'} |x - r_i'|^{\lambda'_i}]$ *satisfies a Lipschitz condition of order* α, *where the* λ'*'s are non-negative constants. The conclusion then becomes* $\rho_2(x)$ *and* $\rho_1(x)$ *are equivalent wherever the polynomials* $[p_n(x; \rho_1)]$ *are bounded and* $\rho(x) \prod_{i=1}^{\nu} |x - r_i|^{\lambda_i} \prod_{i=1}^{\nu'} |x - r_i'|^{\lambda'_i} \neq 0$.

[1] Shohat, J., *Bull. Am. Math. Soc.*, **41**, 51 (1935).

[2] Haar, A., *Math. Ann.*, **78**, 121–136 (1918).

[3] Young, W. H., *Comptes Rendus*, **165**, 696–699 (1917).

[4] Szegö, G., *Math. Zeit.*, **12**, 61–94 (1922).

[5] A general treatment, which sets forth necessary and sufficient conditions for equivalence of expansions in terms of orthogonal functions, is given by Walsh, J. L., and Wiener, N., *Jour. Math. and Phys. of the Mass. Inst. Tech.*, **1**, 103–122 (1922).

[6] For theorems on the order of approximation of continuous functions by polynomials, see Jackson, D., "The Theory of Approximation," *Am. Math. Soc. Colloquium*, **11**, Chap. I.

[7] The results on the boundedness of orthogonal polynomials to be obtained here are sufficient for the present needs. For a more complete, but less simple, treatment, see Bernstein, S., *Jour. Math. pures et appliquées* (9), **9**, 127–177 (1930), and **10**, 219–286 (1931).

[8] See Jackson, D., *Ann. Math.*, **34**, 533 (1933), Theorem I.

[9] Cf. Jackson, loc. cit.,[8] p. 534, where further references are given.

CURVATURE ELEMENT TRANSFORMATIONS WHICH PRESERVE INTEGRABLE FIELDS

BY EDWARD KASNER AND JOHN DE CICCO

DEPARTMENTS OF MATHEMATICS, COLUMBIA UNIVERSITY AND BROOKLYN COLLEGE

Communicated January 11, 1939

I. *Introduction.*—We shall begin by giving some of the fundamental concepts in the geometry of curvature elements of the plane. By a *curvature element*, we mean simply a point of a curve together with the first two derivatives of the curve at the given point. A curvature element is given by the four numbers (x, y, p, q), where (x, y) are the cartesian coördinates of the point, p is the derivative of y with respect to x, that is, $p = dy/dx$, and q is the derivative of y with respect to x, that is, $q =$

$dp/dx = d^2y/dx^2$. We observe that a curvature element may be represented in the plane by a point together with a circular arc passing through that point. We shall call the totality of ∞^4 curvature elements of the plane a *plenum*.

A set of ∞^1 curvature elements of a plane is called a *series* of curvature elements. A series may be given by the three equations $y = y(x)$, $p = p(x)$, $q = q(x)$, where y, p, q are arbitrary functions of x only. If a series satisfies the two conditions $dy - pdx = 0$, $dp - qdx = 0$, the series is termed a *union*. A union obviously consists of a curve or a single fixed lineal element (x_0, y_0, p_0).

A collection of ∞^2 curvature elements is said to be a *field* of curvature elements. A field may be given by the two equations $p = p(x, y)$, $q = q(x, y)$, where p and q are arbitrary functions of (x, y) only. If there exist ∞^1 unions which fit the curvature elements of a given field (that is, all the curvature elements of the ∞^1 unions coincide exactly with all the curvature elements of the given field), the field is said to be an *integrable field*. Of course not all fields are integrable fields. *The necessary and sufficient condition that the field* $\mathrm{p} = \mathrm{p}(\mathrm{x}, \mathrm{y})$, $\mathrm{q} = \mathrm{q}(\mathrm{x}, \mathrm{y})$, *be an integrable field is that the functions* p *and* q *of* (x, y) *satisfy the single partial differential equation of first order* $\mathrm{q} = \mathrm{p}_x + \mathrm{pp}_y$.

A set of ∞^3 curvature elements is termed an *opulence* of curvature elements. An opulence may be given by the single equation $q = q(x, y, p)$, where q is an arbitrary function of (x, y, p) only. An opulence corresponds to a differential equation of the second order. We can always find ∞^2 unions whose curvature elements coincide exactly with the curvature elements of a given opulence. Therefore every opulence is an integrable opulence.

A transformation between the curvature elements of two planes shall be termed a *curvature element transformation*. A curvature element transformation does not, in general, convert every union into a union. If a curvature element transformation converts every union into a union, then it is called a *contact transformation*. It is a well-known theorem of Backlund that *the group of contact transformations of curvature elements is identical with the group of extended contact transformations of lineal elements.*

A curvature element transformation does not, in general, convert every integrable field of curvature elements into an integrable field of curvature elements. The problem of this paper is *to find the group of curvature element transformations which convert every integrable field into an integrable field.* Our fundamental result is that *our group of transformations is the group of contact transformations, that is, the group of extended contact transformations of lineal elements.* Thus the solution of our problem gives us a new characteristic property of the group of extended contact transformations of lineal elements.

We note that we do *not* demand that a single union of an integrable field shall be converted into a single union of the transformed integrable field. But it *does* result from our work that if a transformation carries every integrable field into an integrable field, then a single union of an integrable field is sent into a single union of the transformed integrable field.

II. *Integrable Fields of Curvature Elements into Integrable Fields of Curvature Elements.*—FUNDAMENTAL THEOREM. *The necessary and sufficient conditions that the curvature element transformation*

$X = X(x, y, p, q)$, $Y = Y(x, y, p, q)$, $P = P(x, y, p, q)$, $Q = Q(x, y, p, q)$, *converts every integrable field of curvature elements into an integrable field of curvature elements are that the functions* X, Y, P, Q *satisfy the equations*

$$X = X(x, y, p), \quad Y = Y(x, y, p),$$

$$P = \frac{Y_x + pY_y}{X_x + pX_y} = \frac{Y_p}{X_p}, \tag{L}$$

$$Q = \frac{P_x + pP_y + qP_p}{X_x + pX_y + qX_p},$$

where X *and* Y *are functions of* (x, y, p) *only. Thus our group of transformations is the group of contact transformations of curvature elements. In other words, it is the group of extended contact transformations of lineal elements.*

The sufficiency of our theorem is obvious. The remainder of the paper is concerned with the proof of the necessity of our theorem.

Under our transformation, we must have

$$Q = P_X + PP_Y, \tag{1}$$

whenever $q = p_x + pp_y$. The equation (1) may be written in the form

$$\begin{aligned} Q & \begin{vmatrix} X_x + X_p p_x + X_q q_x & X_y + X_p p_y + X_q q_y \\ Y_x + Y_p p_x + Y_q q_x & Y_y + Y_p p_y + Y_q q_y \end{vmatrix} \\ = & \begin{vmatrix} P_x + P_p p_x + P_q q_x & P_y + P_p p_y + P_q q_y \\ Y_x + Y_p p_x + Y_q q_x & Y_y + Y_p p_y + Y_q q_y \end{vmatrix} \\ + P & \begin{vmatrix} X_x + X_p p_x + X_q q_x & X_y + X_p p_y + X_q q_y \\ P_x + P_p p_x + P_q q_x & P_y + P_p p_y + P_q q_y \end{vmatrix}, \end{aligned} \tag{2}$$

where $p_x = q - pp_y$. The equation (2) must be an identity in p_y, q_x, q_y. Hence, upon setting the coefficients equal to zero, we obtain the *five* partial differential equations of first order

$$
\begin{aligned}
Q\begin{vmatrix} X_q & X_p \\ Y_q & Y_p \end{vmatrix} &= \begin{vmatrix} P_q & P_p \\ Y_q & Y_p \end{vmatrix} + P\begin{vmatrix} X_q & X_p \\ P_q & P_p \end{vmatrix}, \\
Q\begin{vmatrix} X_q & X_y \\ Y_q & Y_y \end{vmatrix} &= \begin{vmatrix} P_q & P_y \\ Y_q & Y_y \end{vmatrix} + P\begin{vmatrix} X_q & X_y \\ P_q & P_y \end{vmatrix}, \\
Q\begin{vmatrix} X_x & X_q \\ Y_x & Y_q \end{vmatrix} &= \begin{vmatrix} P_x & P_q \\ Y_x & Y_q \end{vmatrix} + P\begin{vmatrix} X_x & X_q \\ P_x & P_q \end{vmatrix}, \qquad (3) \\
Q\begin{vmatrix} X_x + pX_y & X_p \\ Y_x + pY_y & Y_p \end{vmatrix} &= \begin{vmatrix} P_x + pP_y & P_p \\ Y_x + pY_y & Y_p \end{vmatrix} + P\begin{vmatrix} X_x + pX_y & X_p \\ P_x + pP_y & P_p \end{vmatrix}, \\
Q\begin{vmatrix} X_x + qX_p & X_y \\ Y_x + qY_p & Y_y \end{vmatrix} &= \begin{vmatrix} P_x + qP_p & P_y \\ Y_x + qY_p & Y_y \end{vmatrix} + P\begin{vmatrix} X_x + qX_p & X_y \\ P_x + qP_p & P_y \end{vmatrix}.
\end{aligned}
$$

In the rest of the paper, we shall concern ourselves with the solution of these complicated equations. We shall show that the only possible solution for the four functions X, Y, P, Q with non-vanishing jacobian is furnished by the equations (L), and therefore we have actually an extended contact transformation.

Now let a, b, c, d represent the quantities

$$a = P_q - QX_q, b = P_p - QX_p, c = P_y - QX_y, d = P_x - QX_x. \quad (4)$$

Before proceeding, it must be observed that *at least one of the four quantities* a, b, c, d *must be different from zero.* For otherwise the jacobian of the transformation would be zero. This is contrary to our hypotheses.

By means of (4), the five equations (3) may be put in the form

$$
\begin{aligned}
&a(Y_p - PX_p) - b(Y_q - PX_q) = 0, \\
&a(Y_y - PX_y) - c(Y_q - PX_q) = 0, \\
&a(Y_x - PX_x) - d(Y_q - PX_q) = 0, \qquad (5) \\
&b[Y_x + pY_y - P(X_x + pX_y)] - (pc + d)(Y_p - PX_p) = 0, \\
&(qb + d)(Y_y - PX_y) - c[Y_x + qY_p - P(X_x + qX_p)] = 0.
\end{aligned}
$$

We shall prove that $Y_q = PX_q$. For let $Y_q \neq PX_q$. Then we can solve the first three of equations (5) for b, c, d, obtaining

$$b = \frac{(Y_p - PX_p)a}{Y_q - PX_q}, c = \frac{(Y_y - PX_y)a}{Y_q - PX_q}, d = \frac{(Y_x - PX_x)a}{Y_q - PX_q}. \quad (6)$$

Upon substituting the values of b, c, d, as given by (6), into the last two of equations (5), we find that they are identically satisfied. Hence under our assumption that $Y_q \neq PX_q$, we find that the equations (6) give the complete solution of the equations (5).

From the above equations, we find that $a \neq 0$. For otherwise all four of the quantities a, b, c, d would be zero. Therefore by the assumption

that $Y_q \neq PX_q$, and by the fact that $a \neq 0$, and by equations (4) and (6), we see that equations (6) are equivalent to the equations

$$\frac{P_q - QX_q}{Y_q - PX_q} = \frac{P_p - QX_p}{Y_p - PX_p} = \frac{P_y - QX_y}{Y_y - PX_y} = \frac{P_x - QX_x}{Y_x - PX_x}. \tag{7}$$

Thus from what was said above, the equations (7) are the complete solution of the equations (5) under the assumption that $Y_q \neq PX_q$.

We observe that the jacobian J of our transformation can be written in the form

$$J = \begin{vmatrix} X_q & X_p & X_y & X_x \\ Y_q & Y_p & Y_y & Y_x \\ P_q & P_p & P_y & P_x \\ Q_q & Q_p & Q_y & Q_x \end{vmatrix} = \begin{vmatrix} X_q & X_p & X_y & X_x \\ Y_q - PX_q & Y_p - PX_p & Y_y - PX_y & Y_x - PX_x \\ P_q - QX_q & P_p - QX_p & P_y - QX_y & P_x - QX_x \\ Q_q & Q_p & Q_y & Q_x \end{vmatrix}. \tag{8}$$

By means of the equations (7), we find that the value of the jacobian J of the transformation, as given by (8), is zero. Thus we have proved our assertion that $Y_q = PX_q$.

Since $Y_q = PX_q$, we find that the equations (5) become

$$\begin{gathered} a(Y_p - PX_p) = 0, a(Y_y - PX_y) = 0, a(Y_x - PX_x) = 0, \\ b[Y_x + pY_y - P(X_x + pX_y)] - (pc + d)(Y_p - PX_p) = 0, \\ (qb + d)(Y_y - PX_y) - c[Y_x + qY_p - P(X_x + qX_p)] = 0. \end{gathered} \tag{9}$$

We shall show that $a = P_q - QX_q = 0$. For if $a \neq 0$, then from what we have just proved and from the first three of the above equations, we obtain

$$Y_q = PX_q, Y_p = PX_p, Y_y = PX_y, Y_x = PX_x. \tag{10}$$

It is easily seen by these equations that the jacobian J of our transformation vanishes. Since this is impossible, the validity of our assertion that $a = 0$ is established.

Since we have proved that $Y_q = PX_q$ and $a = 0$, the equations (5) reduce to the two equations

$$\begin{gathered} b[Y_x + pY_y - P(X_x + pX_y)] - (pc + d)(Y_p - PX_p) = 0, \\ (qb + d)(Y_y - PX_y) - c[Y_x + qY_p - P(X_x + qX_p)] = 0. \end{gathered} \tag{11}$$

We now proceed to discuss the solution of (11). In order to simplify this discussion, we introduce the substitution

$$\lambda = d + pc + qb. \tag{12}$$

Upon eliminating d from (11) and (12), we find that equations (11) are equivalent to the two equations

$$b[Y_x + pY_y + qY_p - P(X_x + pX_y + qX_p)] - \lambda(Y_p - PX_p) = 0,$$
$$c[Y_x + pY_y + qY_p - P(X_x + pX_y + qX_p)] - \lambda(Y_y - PX_y) = 0. \tag{13}$$

We shall prove that $Y_x + pY_y + qY_p - P(X_x + pX_y + qX_p) = 0$. For let us suppose that this is not the case. From the above equations, we find that $\lambda \neq 0$. For otherwise $b = c = 0$, and hence from (12), $b = c = d = 0$. Thus all four of the quantities a, b, c, d are zero. This is impossible.

Now since $Y_x + pY_y + qY_p - P(X_x + pX_y + qX_p) \neq 0$, and $\lambda \neq 0$, we find upon dividing the two equations (13) and making use of (4) that

$$\frac{P_p - QX_p}{Y_p - PX_p} = \frac{P_y - QX_y}{Y_y - PX_y}. \tag{14}$$

By use of (4) and (12), we find from the first of equations (13)

$$\frac{P_p - QX_p}{Y_p - PX_p} = \frac{P_x + pP_y + qP_p - Q(X_x + pX_y + qX_p)}{Y_x + pY_y + qY_p - P(X_x + pX_y + qX_p)}. \tag{15}$$

It follows from (14) and (15) that

$$\frac{P_p - QX_p}{Y_p - PX_p} = \frac{P_y - QX_y}{Y_y - PX_y} = \frac{P_x - QX_x}{Y_x - PX_x}. \tag{16}$$

By these equations, the fact that $Y_q = PX_q$ and the fact that $a = P_q - QX_q = 0$, we find that the jacobian (8) of the transformation is zero. This is impossible. Thus we have proved that

$$Y_x + pY_y + qY_p = P(X_x + pX_y + qX_p).$$

Since $Y_x + pX_y + qY_p = P(X_x + pX_y + qX_p)$, the equations (13) then reduce to the equations

$$\lambda(Y_p - PX_p) = 0, \quad \lambda(Y_y - PX_y) = 0. \tag{17}$$

We shall prove that $\lambda = P_x + pP_y + qP_p - Q(X_x + pX_y + qX_p) = 0$. For if $\lambda \neq 0$, then from the fact that $Y_q = PX_q$, and from what we have just proved above and from the preceding equations (17), we obtain the equations (10). Thus the jacobian of our transformation vanishes so that it follows that $\lambda = 0$.

We have thus far obtained the following solution for our system of equations (3) or (5)

$$\begin{aligned} Y_q &= PX_q, \quad Y_x + pY_y + qY_p = P(X_x + pX_y + qX_p), \\ P_q &= QX_q, \quad P_x + pP_y + qP_p = Q(X_x + pX_y + qX_p). \end{aligned} \tag{18}$$

The equations (18) obviously satisfy the equations (3) or (5) identically. It remains to prove that the only functions X, Y, P, Q with non-vanishing jacobian which satisfy (18) are given by the equations (L) of our theorem.

We shall prove that $X_q = 0$. Let us suppose that $X_q \neq 0$. Then $Y_q \neq 0$ and $P_q \neq 0$. By eliminating P and Q from the equations (18), we find

$$Y_x + pY_y + qY_p = (X_x + pX_y + qX_p)\frac{Y_q}{X_q},$$
$$\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p}\right)\frac{Y_q}{X_q} = \frac{(X_x + pX_y + qX_p)}{X_q}\frac{\partial}{\partial q}\left(\frac{Y_q}{X_q}\right). \tag{19}$$

Upon taking the partial derivative with respect to q of the first of equations (19), and also simplifying the second of equations (19), we obtain

$$(X_qY_p - X_pY_q) + X_q(Y_{xq} + pY_{yq} + qY_{pq}) - Y_q(X_{xq} + pX_{yq} + qX_{pq}) = X_q(X_x + pX_y + qX_p)\frac{\partial}{\partial q}\left(\frac{Y_q}{X_q}\right), \tag{20}$$

$$X_q(Y_{xq} + pY_{yq} + qY_{pq}) - Y_q(X_{xq} + pX_{yq} + qX_{pq}) = X_q(X_x + pX_y + qX_p)\frac{\partial}{\partial q}\left(\frac{Y_q}{X_q}\right).$$

Upon subtracting these two equations, we find

$$\frac{Y_q}{X_q} = \frac{Y_p}{X_p}. \tag{21}$$

Upon substituting (21) into (18), we find that under the assumption that $X_q \neq 0$, the equations (18) are equivalent to

$$P = \frac{Y_q}{X_q} = \frac{Y_p}{X_p},\ Y_x + pY_y = P(X_x + pX_y),$$
$$Q = \frac{P_q}{X_q},\ P_x + pP_y + qP_p = Q(X_x + pX_y + qX_p), \tag{22}$$

where $X_q \neq 0$, $Y_q \neq 0$, $P_q \neq 0$.

Under the existing conditions we see that $X_p \neq 0$ and hence $Y_p \neq 0$. For if $X_p = 0$, then $Y_p = 0$ and hence X and Y are independent of p. From (22), we see that X, Y, P, and Q are independent of p. This makes the jacobian of our transformation zero. This is impossible so that we must have $X_p \neq 0$ and hence $Y_p \neq 0$.

From equations (22) and the fact that $X_p \neq 0$ and $Y_p \neq 0$, we find

$$Y_x + pY_y = (X_x + pX_y)\frac{Y_p}{X_p},$$

$$\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial p}\right)\frac{Y_p}{X_p} = \frac{(X_x + pX_y + qX_p)}{X_q}\frac{\partial}{\partial q}\left(\frac{Y_p}{X_p}\right). \tag{23}$$

Upon taking the partial derivative with respect to p of the first of equations (23), and also simplifying the second of equations (23), we obtain

$$(X_pY_y - X_yY_p) + X_p(Y_{xp} + pY_{yp}) - Y_p(X_{xp} + pX_{yp}) = X_p(X_x + pX_y)\frac{\partial}{\partial p}\left(\frac{Y_p}{X_p}\right), \tag{24}$$

$$X_p(Y_{xp} + pY_{yp}) - Y_p(X_{xp} + pX_{yp}) = X_p(X_x + pX_y)\frac{\partial}{\partial p}\left(\frac{Y_p}{X_p}\right).$$

Upon subtracting these two equations, we find

$$\frac{Y_p}{X_p} = \frac{Y_y}{X_y}. \tag{25}$$

Substituting (25) into (22), we find that under the assumption that $X_q \neq 0$, the equations (18) are equivalent to

$$P = \frac{Y_q}{X_q} = \frac{Y_p}{X_p} = \frac{Y_y}{X_y},\ Y_x = PX_x,$$

$$Q = \frac{P_q}{X_q},\ P_x + pP_y + qP_p = Q(X_x + pX_y + qX_p), \tag{26}$$

where $X_p \neq 0$, $X_q \neq 0$, $Y_p \neq 0$, $Y_q \neq 0$, $P_q \neq 0$. From equations (26), it follows immediately that the jacobian of our transformation is zero. This proves that X_q must be zero.

Since $X_q = 0$, it can be shown that the equations (18) are equivalent to the equations (L) of our theorem. Thus our curvature element transformation must be an extended contact transformation of lineal elements. This completes the proof of our theorem.

An analogous problem in space dealing with the transformation of lineal elements of curves and surfaces is discussed in another paper.[1]

[1] See Kasner, "Lineal Element Transformations Which Convert Normal Families of Curves into Normal Families," *Duke Mathematical Journal*, March (1939). The result gives a new characterization of space contact transformations.

INFORMATION TO CONTRIBUTORS

THE PROCEEDINGS is the official organ of the NATIONAL ACADEMY OF SCIENCES and of the NATIONAL RESEARCH COUNCIL for the publication of brief accounts of important current researches of members of the ACADEMY and of the COUNCIL and of other American investigators. The PROCEEDINGS will aim especially to secure prompt publication of original announcements of discoveries and wide circulation of the results of American research among investigators in other countries and in all branches of science.

ARTICLES should be brief. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, where practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

MANUSCRIPTS should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text-figures of simple character, though more elaborate illustrations may be allowed in special instances to authors willing to pay for their preparation and insertion. Particular attention should be given to arranging tabular matter in a simple and concise manner.

REFERENCES to literature, numbered consecutively, will be placed at the end of the article and short footnotes should be avoided. It is suggested that references to periodicals be furnished in some detail and in general in accordance with the standard adopted for the Subject Catalogue of the International Catalogue of Scientific Literature, viz., name of author, with initials following (ordinarily omitting title of paper), abbreviated name of Journal, with place of publication, series (if any), volume, inclusive pages, year. For example: Montgomery, T. H., *J. Morph., Boston*, **22**, 731–815 (1911); or, Wheeler, W. M., *Konigsburg, Schr. physik., Ges.*, **55**, 1–142 (1914).

PAPERS by members of the ACADEMY may be sent to Edwin Bidwell Wilson, Managing Editor, Harvard School of Public Health, 55 Shattuck St., Boston 17, Mass. Papers by non-members of the ACADEMY or COUNCIL should be submitted through some member.

PROOF will not ordinarily be sent; if an author asks for proof, it will be sent with the understanding that charges for his corrections shall be billed to him. Authors are therefore requested to make final revisions on the typewritten manuscripts. The editors cannot undertake to do more than correct obvious minor errors.

REPRINTS should be ordered at the time of submission of manuscript. They will be furnished to authors at cost, approximately as follows:

SCHEDULE OF RATES FOR REPRINTS, CARRIAGE CHARGES EXTRA

No. copies	50	100	150	200	250
4 pp.	$2.10	$2.40	$2.70	$3.00	$3.30
8 pp.	4.20	4.80	5.40	6.00	6.60
Covers	1.60	2.05	2.50	2.95	3.40

CONTENTS

Page

BOTANY.—GROWTH AND DIFFERENTIATION IN LIVING PLANT MERISTEMS . . *By Edmund W. Sinnott* 55

ANTHROPOLOGY.—QUANTITATIVE ANALYSIS OF THE INTERACTION OF INDIVIDUALS *By Eliot D. Chapple* 58

ASTRONOMY.—ASTROPHYSICAL CONSEQUENCES OF METASTABLE LEVELS IN HYDROGEN AND HELIUM . . . *By O. Struve, K. Wurm and L. G. Henyey* 67

PSYCHOLOGY.—THE RESOLUTION OF SIX TESTS INTO THREE GENERAL FACTORS . *By Edwin B. Wilson and Jane Worcester* 73

PHYSIOLOGY.—TEMPERATURE AND THE CRITICAL INTENSITY FOR RESPONSE TO VISUAL FLICKER. II. . . . . *By W. J. Crozier* 78

PHYSICS.—EXPRESSIONS FOR THE CURRENT IN THE BLOCH APPROXIMATION OF "TIGHT BINDING" FOR METALLIC ELECTRONS . . . . . *By J. Bardeen and J. H. Van Vleck* 82

GEOLOGY.—THE OXIDATION-REDUCTION POTENTIALS OF LAKE WATERS AND THEIR ECOLOGICAL SIGNIFICANCE . . . . . *By G. E. Hutchinson, E. S. Deevey, Jr., and Anne Wollack* 87

MATHEMATICS.—ON IDEALS OF DIFFERENTIAL POLYNOMIALS . . *By J. F. Ritt* 90

MATHEMATICS.—GENERALIZATIONS TO SPACE OF THE CAUCHY AND MORERA THEOREMS . . . . . *By Maxwell Reade and E. F. Beckenbach* 92

MATHEMATICS.—AN EQUIVALENCE THEOREM FOR SERIES OF ORTHOGONAL POLYNOMIALS . . . . . *By G. H. Peebles* 97

MATHEMATICS.—CURVATURE ELEMENT TRANSFORMATIONS WHICH PRESERVE INTEGRABLE FIELDS . . . . *By Edward Kasner and John De Cicco* 104